Bifurcations in two-dimensional differentially heated cavity

Sabiha Aklouche-Benouaguef [1]✉, Belkacem Zeghmati [2]

1 *Laboratoire des Transports Polyphasiques et Milieux Poreux (LTPMP), USTHB, B.P. 32, El-Alia 16111 Alger, Ageria*

2 *Laboratoire de Mathématiques et Physique des Systèmes (LAMPS), Université de Perpignan, 52 Avenue Paul Alduy, 66860 Perpignan, France*

Received 6 May 2016
Revised 24 January 2017
Accepted 2 March 2017
Published online: 5 March 2017

Keywords
Natural convection
Instability
Chaos
Bifurcation
Attractor
Phase trajectory

Abstract: In this work, we propose a numerical analysis of a bidimensional instationary natural convection in a square cavity filled with air and inclined 45 degree versus to horizontal. The vertical walls are subjected to non-uniform temperatures while the horizontal walls are adiabatic. The equations based on the formulation vorticity-stream function are solved using the Alternating Directions Implicit scheme (ADI) and Gauss elimination method. We analyze the influence of Rayleigh number on the roads to chaos borrowed by the natural convection developed in this cavity, and we are looking for stable solutions representing the nonlinear dynamic system. A correlation between the Nusselt number and the Rayleigh number is proposed. We have analyzed the vicinity of the critical point. The transition of the point attractor to another limit cycle attractor is characterized by the Hopf bifurcation.

1. Introduction

The natural convection of Newtonian fluids confined in differentially heated enclosures have attracted considerable attention from researchers in recent years, because of their wide range of applications in thermal engineering, such as solar collectors, electronics, the cooling facilities, the solidification process, and in buildings. The mastery of the appearance of chaotic phenomenon is an interesting and important phase for industrial applications; it can be controlled and especially taken into account in the technical designs. For this, it is interesting to study the chaotic phenomena and effects of various parameters of flow and heat transfer. In general these phenomena are characterized by thermal and hydrodynamic instabilities and the transition is converted into a periodic system. Natural convection induced into a square recesses of which the vertical walls are differentially heated, has been extensively studied and developed (De Vahl Davis 1983; Paolucci and Chenoweth 1989; Ivey 1984; Patterson 1984; Ndamne 1992). A numerical study has been developed by (Wakitani 1997) on the multicellular solutions for a wide range of Rayleigh numbers, and showed that the flow structure is depending on the initial conditions. Other researchers are interested in exploring the routes to chaos and in particular, the temporal sequence of bifurcation to chaos Bifurcation method is a tool for numerical analysis of the stability of dynamic systems and nonlinear effects. These methods were introduced by Jahnke and Culick (2011). The nonlinear behavior can best be understood in terms of bifurcation of the dynamic system.

This dynamic system represents all critical points where balances undergo a change in their stability. Many roads to chaos in natural convection were found theoretically and experimentally (Manneville and Pomeau 1980; Behringer 1985; Feigenbaum 1980; Mukutmoni and Yang 1993a; Bratsun et al. 2003; Mukutmoni and Yang 1993b). The case of natural convection in a cavity was developed by D'Orazio et al. (2004) who have used the algorithm SIMPLE-C to solve the equations of energy, movement, and continuity. Studies of the effect of the various values of aspect ratio have detected the presence of either a regular cell, or two regular cells, or two periodic cells, or finally three periodic cells. At each bifurcation abrupt, changes of the Nusselt number were observed. Works of (Mizushima and Hara 2000) on the horizontal and vertical cavities have shown that for the horizontal cavity, thermal convection occurs over a number of critical Rayleigh due to instability, while for the vertical cavity heated from a wall, natural convection occurs for small Rayleigh numbers and has a unicellular global circulation. Exploring multicellular convection when the vertical cavity is gradually inclined from the horizontal plane was examined by an analysis of bifurcations. Finally, a numerical investigation on unsteady natural convection was devoted to the multiplicity of solutions for an aspect ratio of cavity equal to 1 (Aklouche-Benouaguef et al. 2014). The different stable solutions obtained were represented by attractors in

✉ Corresponding author. E-mail address: mesadz@yahoo.fr

phase spaces. These attractors were characterized by their fractal dimension. The study highlighted the scenario to chaos and also explains the divergence of two very close initial solutions by determining the Lyapunov exponent. The objective of this present work is to show that for a tilted square cavity, a dynamic system leads to a deterministic chaos. We analyze the vicinity of the first critical point for looking for different solutions representing the stable dynamic system, and attempting to establish a correlation between the number of Nusselt and Rayleigh number.

2. Mathematical Formulation

2.1 Mathematical model

Figure 1 shows the two-dimensional geometry of a square cavity filled with air and inclined (45 degrees). The horizontal walls are adiabatic and the vertical walls are brought to non-uniform temperatures: the lower half portion of the wall is hotter than the upper half portion. The fluid properties are assumed to be constant except for the density in the term of the thrust Boussinesq approximation. The two-dimensional flow is considered laminar while the viscous dissipation in the energy equation and thermal radiation between the walls is negligible.

2.1.1 Adimensional Equations

The scaling height, time, and velocity are set dimensioneless *H*, *H / a*^{2}*, and H / a*, respectively. The dimensionless equations of the stream function, vorticity and energy equation are written as follows.

Vorticity equation

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=-\Omega \quad (1)$$

Momentum equation

$$\frac{d\Omega}{dt}+\frac{\partial}{\partial x}\left(U.\Omega-\Pr\frac{\partial\Omega}{\partial x}\right)+\frac{\partial}{\partial y}\left(V.\Omega-\Pr\frac{\partial\Omega}{\partial y}\right)=Ra.\Pr.\left[\cos\alpha\left(\frac{\partial T}{\partial x}\right)-\sin\alpha\left(\frac{\partial T}{\partial y}\right)\right] \quad (2)$$

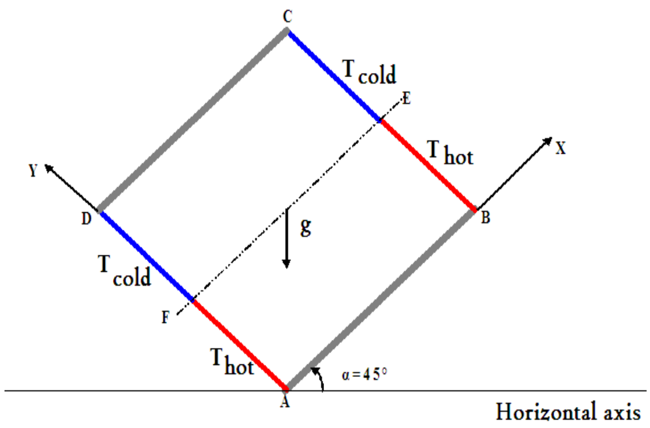

Fig .1. Physical system.

Heat transfer equation

$$\frac{dT}{dt}+\frac{\partial}{\partial x}\left(U.T-\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(V.T-\frac{\partial T}{\partial y}\right)=0 \quad (3)$$

where

$$x=x'/H,\ y=y'/H,\ u=u'H/a,\ v=v'H/a,$$
$$\Omega=\Omega'H^2/a,\ t=t'a/H^2,\ \Psi=\Psi'/a,\ T=\left(T'-T_f\right)\left(T_c-T_f\right) \quad (4)$$

2.2. Initial and boundary conditions

Initial conditions: For $t<t_0$, t_0 being the time from which the vertical walls are subjected to non uniform temperatures.

$$u=v=\psi=\Omega=0,\quad T=0 \quad (5)$$

Boundary conditions:

for $t\geq t_0$ vertical walls : $x=0$ and $x=1$

$$0<y<1:\ u=v=\psi=0\ ,\ \Omega=-\frac{\partial^2\Psi}{\partial x^2} \quad (6)$$

$$0<y<1/2:\quad T=1$$
$$1/2<y<1:\quad T=0 \quad (7)$$
$$y=1/2:\quad T\left(x,1/2\right)=0.5$$

Horizontals walls: $y=0$ and $y=1$ and $0<y<1$

$$u=v=\psi=0\ ,\ \frac{\partial T}{\partial y}=0 \quad (8)$$

2.3. Nusselt number

The global Hot Nusselt number (Nu_{hot}) and the global Cold Nusselt number (Nu_{cold}) are defined by:

$$Nu_{Hot}=\int_0^{1/2}\frac{\left(-\partial T/\partial x\right)_{x=0}y}{T_p-T_m}dy+\int_0^{1/2}\frac{\left(\partial T/\partial x\right)_{x=1}y}{T_p-T_m}dy \quad (9)$$

$$Nu_{Cold}=\int_{1/2}^{1}\frac{\left(\partial T/\partial x\right)_{x=0}y}{T_p-T_m}dy+\int_{1/2}^{1}-\frac{\left(\partial T/\partial x\right)_{x=1}y}{T_p-T_m}dy \quad (10)$$

T_m is the average temperature of the fluid in the cavity. Its expression is:

$$T_m=\frac{\sum_{i,j}^{Nx,Ny}T(i,j)}{N_x\,N_y} \quad (11)$$

N_x and N_y are the node numbers along [ox) and [oy) axis

2.4 Numerical Methodology

The discretization, using the alternating directions implicit scheme (ADI) of equations (2) and (3) associated to the boundary conditions (6-8), leads to algebraic equations systems which can be written as tridiagonal matrices. These systems are solved by Gauss elimination method and an iterative procedure. The vorticity equation is solved by an implicit numerical scheme and a successive over relaxation method. For each time step, the convergence is assumed to be reached for the k-th iteration when the criterion defined by (12) is verified.

$$\max_{p=1}^{3}\frac{\sum_i\sum_j\left|f_{i,j}^{k+1}-f_{i,j}^{k}\right|}{\sum_i\sum_j\left|f_{i,j}^{k+1}\right|}\le epsilon \qquad (12)$$

where epsilon is less than or equal to 10^{-5} for the Poisson equation (1) and less than or equal to 10^{-6} for the equations of vorticity and energy. The expression of the frontiers vorticity is inferred from Woods (1954) approximations:

$$\Omega_p = -\frac{1}{2}\,\Omega_p+1-\,3\,(\Psi_p+1-\Psi_p)\,/\Delta n^2 \qquad (13)$$

Where *p* is the wall and Δn is the space step along the normal to the wall. The average Nusselt numbers were computed using the Simpson rule of the local Nusselt number.

3. Validation and Results

3.1 Validation

We validated our numerical code with the benchmark solution of De Vahl Davis (1983) and experimental results of Ndamne (1992). Our results are in good agreement with those of De Vahl Davis (1983). In addition, the discrepancies between our results and experimental results of Ndamne (1992) (Figure 2) are inferior to 0.6% for the temperature. Our numerical code is validated. A grid sensibility analysis has been also performed. The sensitivity to the mesh was made by considering the mesh that has the smallest relative error.

3.2. Non linear analysis

The stationary solution is obtained for a Rayleigh number equal to 10^5. This solution is represented in a phase space, a geometrical object called gravity point limit (Figure 3). This attractor is a spiral that tends to a fixed point, reflecting the regression of the effects of nonlinearity. The time evolution of this endpoint is characterized by a damping of the amplitudes over time (Figure 4).

The Fourier spectrum for the stationary solution shows no frequency. The geometric shape of attractor is changed for a critical Rayleigh number equal to 3.994 10^5. It represents the first

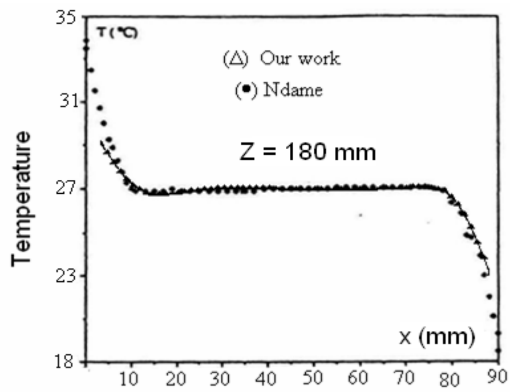

Fig.2. Validation with Ndamne (1992).

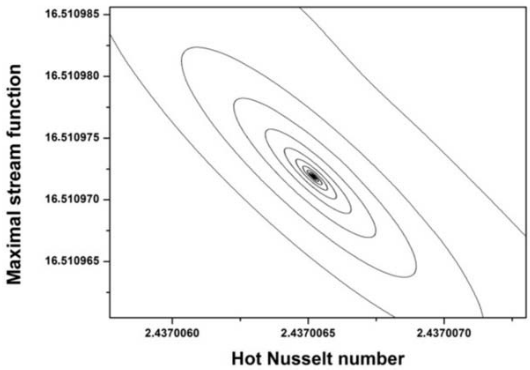

Fig.3. Attractor: Limit Point.

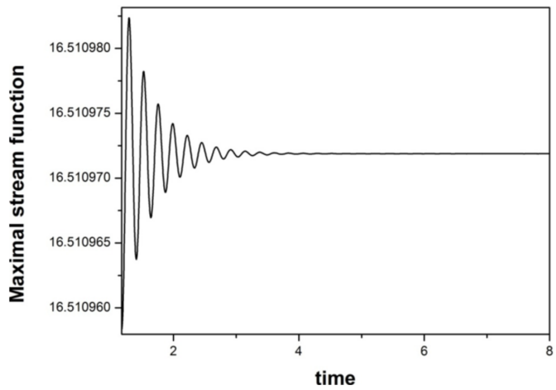

Fig.4. Temporal evolution: Stationary solution.

bifurcation from the steady state to the oscillatory state. To characterize the first bifurcation, it is necessary to conduct a study in the vicinity of the critical point. For this purpose, we plotted the curves showing the variation of the amplitude of a dynamic parameter as a function of Rayleigh number, then depending on the amount of $(Ra-Ra_{cr})^{1/2}$. Figure 5 shows the evolution of the square of the amplitude of the hot Nusselt number and the stream function with the Rayleigh number. These curves show that the separation point corresponds to the critical Rayleigh number obtained graphically. This allows us to assess the relative error between the value found from code and from the plot graph is very small, of the order 0.062%. The curve (dashed lines) is the amplitude ($Amplitude^2$) of the signal obtained from the Fourier transform of the temporal evolution of the stream function depending on the Rayleigh number. It is the same for the curve (solid lines) is the signal amplitude of the temporal evolution of the hot Nusselt number.

Figure 6 shows the linear development of the amplitude of a dynamic parameter such as the Nusselt number as function of the difference $(Ra-Ra_{cr})^{1/2}$. This linearity characterizes the nature of the first bifurcation which is a Hopf bifurcation. The results show that the periodic system settles for a Rayleigh number equal to 3.994. 10^5. We represent the spectrum amplitude of the oscillatory solution (Figure 7) and the attractor limit cycle (Figure 8). The birth of the limit cycle is confirmed in the power spectrum by the appearance of an energy frequency that is relatively zero amplitude. This is consistent with results from the literature (Berge et al. 1998).

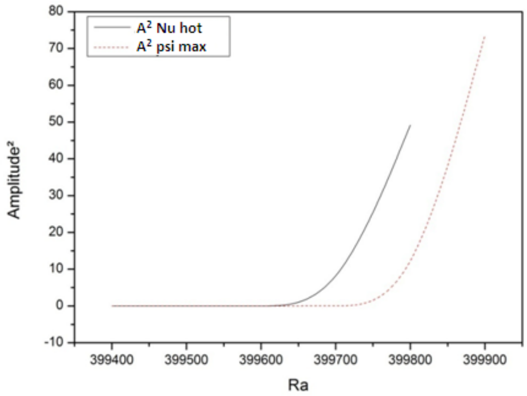

Fig.5. Graphic determination of critical Rayleigh number

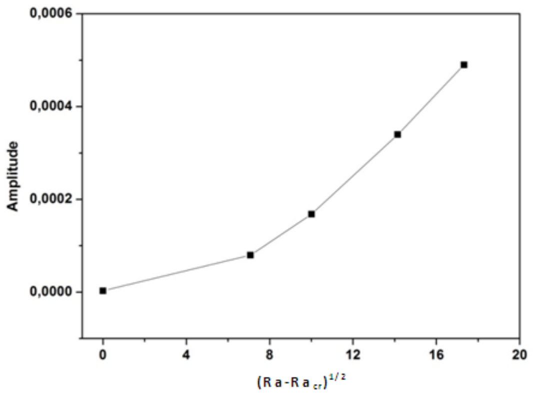

Fig.6. Amplitude spectrum of the Nusselt number as a function of $(Ra\text{-}Ra_{cr})^{1/2}$.

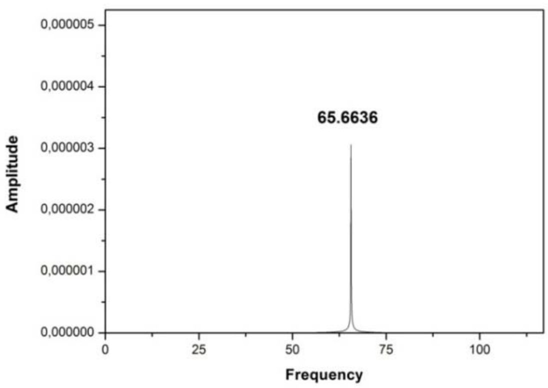

Fig.7. Amplitude spectrum: Periodic solution.

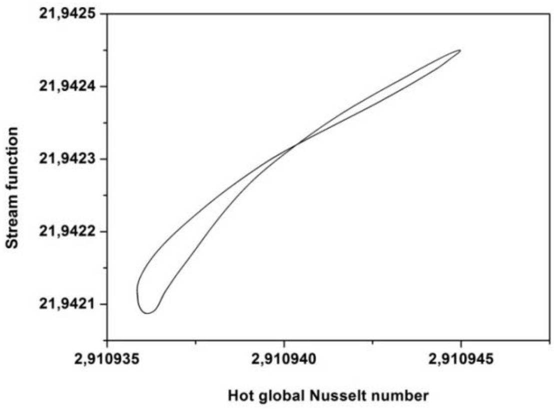

Fig.8. Attractor: Limit Cycle.

The frequencies of the stationary solution are obtained using the fast Fourier transform with a spectral resolution of less than 0.2. It appears that the frequencies become sensitive to space and time when the Rayleigh number increases, although the amplitudes are less sensitive than the frequencies. The first fork being characterized, the other bifurcations are determined by the research of stable solutions in non-stationary.

The calculations helped to highlight six stable solutions corresponding to the Rayleigh numbers: 4.2×10^5, 4.3×10^5, 4.4×10^5, 4.7×10^5, 4.8×10^5 and $4.9\text{x}10^5$. These Rayleigh numbers were determined by studying the sensitivity of the frequency to mesh and time step. The mesh from which the frequency is stable is considered. It is the same for the time step. We represent the phase portrait (Figure 9) and the amplitude spectrum (Figure 10) for the periodic stable solution corresponding to the Rayleigh number equal to 4.9×10^5.

The stationary and periodic solutions are precocious compared to the horizontal cavity (result compared to the previous working (Aklouche-Benouaguef et al. (2014)). Intensified natural convection to a Rayleigh number Ra = 3×10^6 leads to chaos. The figure 11 shows the appearance of chaos.

3.3 Correlation between the Nusselt number and the Rayleigh number

Figure 12 shows in logarithmic representation, the evolution of the Nusselt number of hot depending on the Rayleigh number.

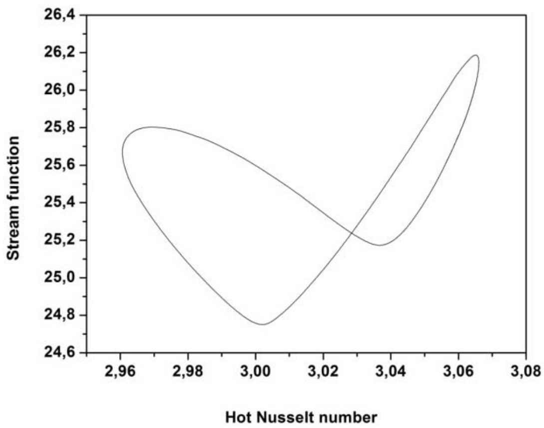

Fig.9. Temporal Phase trajectory, Ra=$4.9\text{x}10^5$.

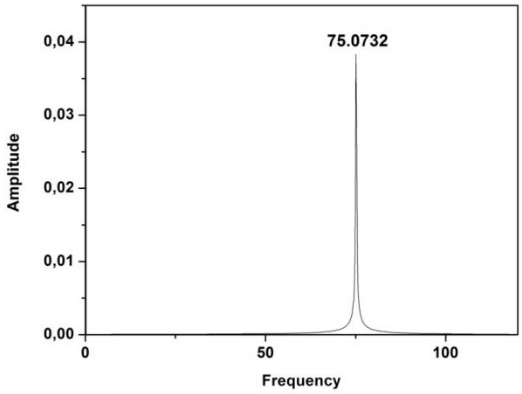

Fig.10. Amplitude spectrum, Ra=$4.9\text{x}10^5$.

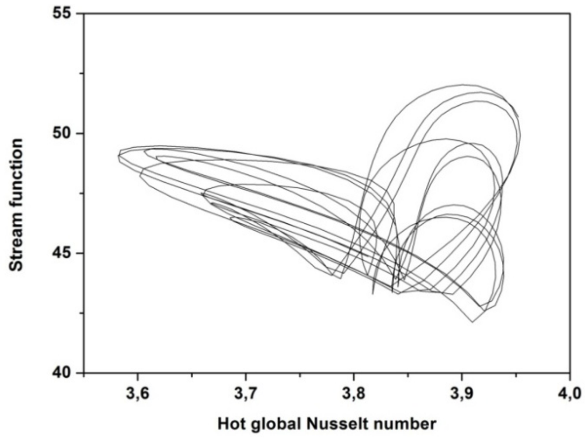

Fig.11. Appearance of chaos, Ra=2x10^6.

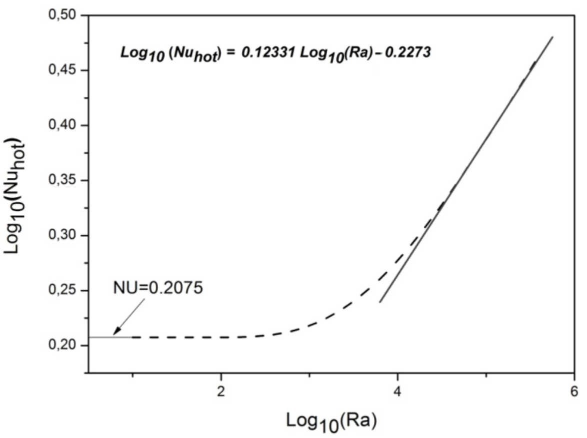

Fig.12. Log(Nu $_{hot}$)= log (Ra).

The resulting curve presents two parts: A portion where the regime is conductive with: Nu_{hot} ~ 0.2075. The second part, relating to the convective regime, an upward space can be correlated by the following expression Nu_{hot}= 0.5925×$Ra^{0.1233}$.

4. Conclusion

A numerical study was conducted on natural convection in a square cavity containing air and inclined at an angle α = 45. The developed computer code was validated with some theoretical and experimental results published in the literature. A stability study was done. The search of the mesh and time for each Rayleigh number led to the determination of the optimum mesh size and time step. The stationary solution, represented by a spiral, characterizes the regression of the effects of non-linearity of the equations representing physical system. This study showed that the flow in the cavity undergoes bifurcation sequences of the stationary state to a convective oscillatory state. The study in the vicinity of the critical point has allowed characterizing the nature of the first Hopf bifurcation. Stable solutions are unsteady in the nonlinear dynamic system. These are the solutions that are useful for calculating the fractal dimension of the attractor. The stationary and periodic solutions are precocious compared to the horizontal cavity (result compared to the previous working (Aklouche-Benouaguef et al. (2014)). This system undergoes a deterministic chaos. The roads used by natural convection in the cavity move towards the scenario of chaos which is probably almost periodicity

References

Aklouche-Benouaguef, S., B. Zeghmati, K. Bouhadef (2014) Numerical simulation of chaotic natural convection in a differentiated closed square cavity. Numerical Heat Transfer, Part A 65(3): 229-246.

Behringer, R.P. (1985) Rayleigh-Benard convection and turbulence in liquid helium. Reviews of modern physics 57(3): 657.

Berge, P., Y. Pomeau & C. Vidal (1998) L'ordre dans le chaos : Vers une Approche Déterministe de la Turbulence. Paris: Hermann.

Bratsun, D. A., A. V. Zyuzgin & G. F. Putin (2003) Non-linear dynamics and pattern formation in a vertical fluid layer heated from the side. International journal of heat and fluid flow 24(6): 835-852.

D'Orazio, M.C.,C. Cianfrini & M. Corcione (2004) Rayleigh Benard convection in tall rectangular enclosures. International Journal Thermal Sciences 43(2): 136-144.

De Vahl Davis, G. (1983) Natural Convection of air in a square Cavity: A Benchmark Numerical Solution. International Journal for Numerical Methods in Fluids 3(3): 249-264.

Feigenbaum, M.J. (1980) The transition to aperiodic behavior in turbulent systems. Communications in mathematical physics 77(1): 65-86.

Ivey, G. N. (1984) Experiments on transient natural convection in a cavity. Journal of Fluid Mechanics 144: 389-401.

Jahnke, C., & F.E.C. Culick (1994). Application of bifurcation theory to the high-angle-of-attack dynamics of the F-14. Journal of Aircraft, 31(1), 26-34.

Manneville, P. & Y. Pomeau (1980) Different ways to turbulence in dissipative dynamical systems. Physica D: Nonlinear Phenomena, 1(2): 219-226.

Mizushima, J. & Y. Hara (2000) Routes to unicellular convection in a tilted rectangular cavity. Journal of the Physical Society of Japan, 69(8): 2371-2374.

Mukutmoni, D.D. & K.T. Yang (1993a) Rayleigh-Bénard Convection in a Small Aspect Ratio Enclosure: Part I-Bifurcation to Oscillatory Convection. Journal of Heat Transfer 115(2):360-366.

Mukutmoni, D. & K.T. Yang (1993b) Rayleigh- Bénard Convection in a Small Aspect Ratio Enclosure: Part II-Bifurcation to Chaos. Journal of Heat Transfer 115(2):367-375.

Ndamne, A. (1992) Etude expérimentale de la convection naturelle en cavité: de l'état stationnaire au chaos (Doctoral dissertation).

Paolucci, S., & D.R. Chenoweth (1989) Transition to chaos in a differentially heated vertical cavity. Journal of Fluid Mechanics 201 : 379-410.

Patterson, J.C. (1984) On the existence of an oscillatory approach to steady natural convection in cavities. Journal of heat transfer 106(1) : 104-108.

Wakitani, S. (1997). Development of multicellular solutions in natural convection in an air-filled vertical cavity. Journal of heat transfer. 119(1): 97-101.

Woods, L.C. (1954). A note on the numerical solution of fourth order differential equations. The Aeronautical Quarterly 5(4) : 176-184.