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A Log-Probability-Weighted-Moments type estimator for the extreme value index in a truncation scheme.

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ABSTRACT: The limit theorems of asymptotic behavior of tail index estimators for right truncation Pareto-like data requires some regularity assumptions either on tail indices ($\gamma_1 < \gamma_2$) or on the dependence structure condition between the truncation variable and the interest one. In this paper, we introduce a new estimator for the tail index based on the Log-Probability-Weighted-Moments method and, getting rid of aforementioned assumptions, we establish its consistency and asymptotic normality. We show, by simulation, that the newly proposed estimator behaves well both in terms of bias and mean squared error.

Keywords: Empirical process, Extreme value index, Product-limit estimator, Truncated data.

MSC: Primary 62G32, 62G30, Secondary 60G70, 60F17.

1 Introduction

Let (X_i^*, Y_i^*), $i = 1, ..., N \geq 1$ be a sample from a couple (X^*, Y^*) of independent positive random variables (rv's) defined over some probability space $(\Omega, \mathcal{A}, \mathbf{P})$, with continuous distribution functions (df's) F^* and G^* respectively. Suppose that X^* is right-truncated by Y^*, in the sense that X_i^* is only observed when $X_i^* \leq Y_i^*$. Throughout the paper, we will use the notation $\overline{\mathcal{S}}(x) := \mathcal{S}(\infty) - \mathcal{S}(x)$, for any $\mathcal{S}$. We assume that both right-tail functions $\overline{F}^*$ and $\overline{G}^*$ are regularly varying at infinity with respective tail indices $-1/\gamma_1$ and $-1/\gamma_2$, notation: $\overline{F}^* \in \mathcal{RV}_{(-1/\gamma_1)}$ and $\overline{G}^* \in \mathcal{RV}_{(-1/\gamma_2)}$. That is, for any $s > 0$

$$\frac{\overline{F}^*(st)}{\overline{F}^*(t)} \to s^{-1/\gamma_1} \text{ and } \frac{\overline{G}^*(st)}{\overline{G}^*(t)} \to s^{-1/\gamma_2}, \text{ as } t \to \infty. \tag{1.1}$$

Let us now denote (X_i, Y_i), $i = 1, ..., n$, to be the observed data, as copies of a couple of rv's (X, Y) with joint df $\mathcal{T}$, corresponding to the truncated sample (X_i^*, Y_i^*), $i = 1, ..., N$, where $n = n_N$ is a sequence of discrete rv's. By the strong law of the large numbers, we have

$$n_N/N \to \mathbf{P}(X^* \leq Y^*) = \int_0^\infty \overline{G}^*(z)\, dF^*(z) =: p, \tag{1.2}$$

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as $N \to \infty$, almost surely (a.s.), where p stands for the percentage of the observed data. This property allows us to assume, without loss of generality, that for any subsequence a_n of n, we may drop "a.s." in the strong limit $a_n \dot{\to} a \leq \infty$ as $N \to \infty$. For $x, y \geq 0$, we have

$$\mathcal{T}(x,y) := p^{-1} \int_0^y F^*(\min(x,z))\, dG^*(z),$$

having (marginal) right-tails

$$\overline{F}(x) = -p^{-1} \int_x^\infty \overline{G}^*(z)\, d\overline{F}^*(z) \text{ and } \overline{G}(y) = -p^{-1} \int_y^\infty F^*(z)\, d\overline{G}^*(z).$$

Note that $\overline{F} \in \mathcal{RV}_{(-1/\gamma)}$ and $\overline{G} \in \mathcal{RV}_{(-1/\gamma_2)}$, where $\gamma := \gamma_1\gamma_2/(\gamma_1 + \gamma_2)$ (see, e.g., [7]). Motivated by an application to real dataset of lifetimes of automobile brake pads ([13], page 69), recently [7] introduced an estimator of γ_1 defined by

$$\widehat{\gamma}_1(k_1, k_2) := \frac{\widehat{\gamma}_2(k_2)\,\widehat{\gamma}(k_1)}{\widehat{\gamma}_2(k_2) - \widehat{\gamma}(k_1)},$$

where $k_1 = k_1(n)$ and $k_2 = k_2(n)$ are two distinct sample fractions used, respectively, in Hill's estimators ([10])

$$\widehat{\gamma}(k_1) := \frac{1}{k_1} \sum_{i=1}^{k_1} \log \frac{X_{n-i+1:n}}{X_{n-k_1:n}} \text{ and } \widehat{\gamma}_2(k_2) := \frac{1}{k_2} \sum_{i=1}^{k_2} \log \frac{Y_{n-i+1:n}}{Y_{n-k_2:n}},$$

of tail indices γ and γ_2, with $X_{1:n} \leq ... \leq X_{n:n}$ and $Y_{1:n} \leq ... \leq Y_{n:n}$ being the order statistics pertaining to the samples $(X_1, ..., X_n)$ and $(Y_1, ..., Y_n)$ respectively. [2] considered a single sample fraction $k = k_1 = k_2$ satisfying $1 < k < n$, $k \to \infty$ and $k/n \to 0$, as $N \to \infty$, and defined the corresponding estimators of γ, γ_2 and γ_1 by

$$\widehat{\gamma} := \frac{1}{k} \sum_{i=1}^{k} \log \frac{X_{n-i+1:n}}{X_{n-k:n}},\ \widehat{\gamma}_2 := \frac{1}{k} \sum_{i=1}^{k} \log \frac{Y_{n-i+1:n}}{Y_{n-k:n}},$$

and

$$\widehat{\gamma}_1^{(\mathbf{GS})} := \frac{\frac{1}{k} \sum_{i=1}^{k} \sum_{j=1}^{k} \log \frac{X_{n-i+1:n}}{X_{n-k:n}} \log \frac{Y_{n-j+1:n}}{Y_{n-k:n}}}{\sum_{i=1}^{k} \log \frac{Y_{n-i+1:n} X_{n-k:n}}{Y_{n-k:n} X_{n-i+1:n}}}.$$

Assuming regular variation conditions (1.1) and the tail dependence assumption (see, e.g., [16]) , they also provided a Gaussian representation is terms of a two-parameter Wiener process which leads to asymptotic normality of $\widehat{\gamma}_1^{(\mathbf{GS})}$. More recently, [3] proposed a new estimation method based on the product-limit estimator of underlying df F^*, to derive the following estimator

$$\widehat{\gamma}_1^{(\mathbf{BMN})} := \left(\sum_{i=1}^{k} \frac{F_n^*(X_{n-i+1:n})}{C_n(X_{n-i+1:n})} \right)^{-1} \sum_{i=1}^{k} \frac{F_n^*(X_{n-i+1:n})}{C_n(X_{n-i+1:n})} \log \frac{X_{n-i+1:n}}{X_{n-k:n}},$$

where

$$F_n^*(x) := \prod_{i:X_i > x} \exp\left\{ -\frac{1}{nC_n(X_i)} \right\},$$

is the so-called product-limit Woodroofe's estimator [18] of df F^* and $C_n(x) := n^{-1} \sum_{i=1}^{n} \mathbb{I}_{\{X_i \leq x \leq Y_i\}}$, where $\mathbb{I}_A$ stands for the indicator function of set A. The authors also established the consistency and asymptotic normality of their estimator but by considering only the case $\gamma_1 < \gamma_2$. More precisely

$$\sqrt{k}\left(\widehat{\gamma}_1^{(\mathbf{BMN})} - \gamma_1 \right) \xrightarrow{\mathcal{D}} \mathcal{N}(\mu, \sigma_1^2), \text{ as } n \to \infty,$$

where

$$\sigma_1^2 := \gamma^2 (1 + (\gamma_1/\gamma_2)) \left(1 + (\gamma_1/\gamma_2)^2\right) / (1 - (\gamma_1/\gamma_2)).$$

The bias reduction of this estimator was addressed in [4], [3], [11] and more recently in [14]. For their part, [19] proposed a similar estimator to $\widehat{\gamma}_1^{(\mathbf{BMN})}$ (with deterministic threshold) and established its asymptotic normality by assuming condition $\gamma_1 < \gamma_2$ as well. Although this condition seems reasonable, it is better that it is not imposed. In conclusion, as mentioned above, the asymptotic behavior of the already proposed estimators was studied either by making restriction of tail indices or by assuming the tail dependence condition between the truncation and truncated rv's. To get rid of these assumptions, we propose an alternative estimation method that we next give their details.

1.1 New estimator for the tail index γ_1

We have already noticed that both two estimators of the tail index γ_1, given by [3] and [19], are based on the nonparametric product-limit estimators of the underlying df F^*. Although , this approach provides good estimators in terms of bias and the root mean squared error (rmse), their corresponding consistency and asymptotic normality are valid only for Pareto-type models satisfying assumption $\gamma_1 < \gamma_2$. Then our main goal is to define an estimator for γ_1 that works for both $\gamma_1 < \gamma_2$ and $\gamma_1 \geq \gamma_2$. To this end, we introduce a new estimation method inspired by the log probability weighted moments (LPWM) estimation method, for complete data, given recently by [5]. Let us define the following ratio of tail expectations

$$L_t(r,s) := \frac{\mathbf{E}\left[\left(\overline{G}(X)\right)^r (\log(X/t))^s \mid X > t\right]}{\mathbf{E}\left[\left(\overline{G}(X)\right)^r \mid X > t\right]},\ r,s \geq 0,\ t > 0.$$

For suitable values of r and s with large t, the ratio $L_t(r,s)$ serve us to estimate the tail indices $(\gamma, \beta, \gamma_2)$ and also the second-order parameters (ρ_F, ρ_H, ρ_G) , given in (2.1) and (2.2) , which is out of scope of the paper. Indeed, we showed in Proposition 6.1, that

$$L_t(r,s) \to \left(\frac{\gamma_1\gamma}{(1+r)\gamma_1 - r\gamma}\right)^s \Gamma(s+1),\ \text{as } t \to \infty,$$

where $\Gamma : z \to \int_0^\infty x^{z-1} e^{-x} dx$, $z > 0$, is the usual gamma function. In particular, we have

$$\gamma_t := L_t(0,1) = \frac{\int_t^\infty \log(x/t)\, dF(x)}{\overline{F}(t)} \to \gamma,\ \text{as } t \to \infty$$

and

$$\beta_t := L_t(1,1) = \frac{\int_t^\infty \overline{G}(x) \log(x/t)\, dF(x)}{\int_t^\infty \overline{G}(x)\, dF(x)} \to \beta,\ \text{as } t \to \infty,$$

where

$$\beta := \frac{\gamma_1\gamma}{2\gamma_1 - \gamma} = \frac{\gamma_1\gamma_2}{2\gamma_1 + \gamma_2}.$$

This mean that

$$\overline{H}(x) := \frac{\int_x^\infty \overline{G}(x)\, dF(x)}{\int_0^\infty \overline{G}(x)\, dF(x)}$$

is regularly varying with at infinity with tail index $-1/\beta$. It is clear that the above $\beta-$formula, implies that

$$\gamma_1 = \frac{\beta\gamma}{2\beta - \gamma},$$

which will used to estimate γ_1 by means of Hill's estimators $\widehat{\gamma}$ and $\widehat{\beta}$ that will be defined below. To this end, let us $t = X_{n-k:n}$ and then replace, in β_t above, both F and G by their respective empirical df's

$$F_n(x) := n^{-1} \sum_{i=1}^n \mathbb{I}_{\{X_i \leq x\}} \text{ and } G_n(y) := n^{-1} \sum_{i=1}^n \mathbb{I}_{\{Y_i \leq y\}},$$

to get

$$\frac{\int_{X_{n-k:n}}^\infty \overline{G}_n(x) \log(x/X_{n-k:n})\, dF_n(x)}{\int_{X_{n-k:n}}^\infty \overline{G}_n(x)\, dF_n(x)},$$

which equals

$$\widehat{\beta} := \sum_{i=1}^{k} c_{i,n} \log \left(X_{n-i+1:n} / X_{n-k:n} \right),$$

where

$$c_{i,n} := \frac{\overline{G}_n \left(X_{n-i+1:n} \right)}{\sum_{i=1}^{k} \overline{G}_n \left(X_{n-i+1:n} \right)}.$$

Finally, by using the above formula of γ_1, we end up with a new estimator for γ_1 as follows

$$\widehat{\gamma}_1 := \frac{\widehat{\beta}\widehat{\gamma}}{2\widehat{\beta} - \widehat{\gamma}}. \tag{1.3}$$

To establish the consistency and asymptotic normality of $\widehat{\gamma}_1$, we will make use the tail empirical process technics given in [9], which is used recently by [4] in the truncation case. The tail empirical process corresponding to df F, by

$$D_n^{(1)}(x) := M_n^{(1)}(x) - r_1(x), \text{ for } x \geq 1,$$

where

$$M_n^{(1)}(x) := \frac{\int_{xX_{n-k:n}}^{\infty} dF_n(w)}{\int_{X_{n-k:n}}^{\infty} dF_n(w)} = \frac{n}{k} \overline{F}_n \left(X_{n-k:n} x \right) \text{ and } r_1(x) := x^{-1/\gamma},$$

so that

$$\widehat{\gamma} - \gamma = \int_1^{\infty} x^{-1} D_n^{(1)}(x) \, dx.$$

Likewise, we define the tail empirical process corresponding to df H, by

$$D_n^{(2)}(x) := M_n^{(2)}(x) - r_2(x), \ 2, \text{ for } x \geq 1,$$

where

$$M_n^{(2)}(x) := \frac{\overline{H}_n \left(xX_{n-k:n} \right)}{\overline{H}_n \left(X_{n-k:n} \right)} = \frac{\int_{xX_{n-k:n}}^{\infty} \overline{G}_n(w) \, dF_n(w)}{\int_{X_{n-k:n}}^{\infty} \overline{G}_n(w) \, dF_n(w)} \text{ and } r_2(x) := x^{-1/\beta},$$

so that

$$\widehat{\beta} - \beta = \int_1^{\infty} x^{-1} D_n^{(2)}(x) \, dx.$$

By using formula (1.3), we get

$$\widehat{\gamma}_1 - \gamma_1 = \int_1^{\infty} x^{-1} \left(c_{n1} D_n^{(1)}(x) - c_{n2} D_n^{(2)}(x) \right) dx, \tag{1.4}$$

where

$$c_{n1} := \frac{2\beta^2}{\left(\widehat{\gamma} - 2\beta \right) \left(\gamma - 2\beta \right)} \text{ and } c_{n2} := \frac{\widehat{\gamma}^2}{\left(\widehat{\gamma} - 2\widehat{\beta} \right) \left(\widehat{\gamma} - 2\beta \right)}.$$

By means of previous functional representations and weak approximations corresponding to $D_n^{(1)}(x)$ and $D_n^{(2)}(x)$ below, we establish both consistency and asymptotic normality of $\widehat{\gamma}_1$.The rest of the paper is organized as follows. In Section 2, we state our main results, namely consistency and asymptotic normality of $\widehat{\gamma}_1$. A simulation study is carried out, in Section 3, to illustrate the performance of $\widehat{\gamma}_1$. The proofs are postponed to Appendix 5 whereas some results that are instrumental to our needs are gathered in the Appendix 6.

2 MAIN RESULTS

Next we need to the usual second-order condition that specify the rate of convergence of regular variation functions. More precisely for a given function $\varphi \in \mathcal{RV}_{(-1/\alpha)}$, we assume that

$$\frac{1}{A_{\varphi}(t)}\left(\frac{\varphi(tx)}{\varphi(t)} - x^{-1/\alpha}\right) \to x^{-1/\alpha}\frac{x^{\tau/\alpha}-1}{\tau\alpha},\ \text{for } x>0,$$

where $|A_{\varphi}|$ is regularly varying (at infinity) with tail index (second-order parameter) $\tau/\alpha < 0$ [**?**, see, e.g.,]]deHS96. A function φ satisfying this condition is denoted $\varphi \in 2\mathcal{RV}_{(-1/\alpha)}(A_{\varphi},\tau)$. For convenience, we set $\mathbb{A}_{\varphi} := A_{\varphi} \circ \mathbb{U}_F$, where $\mathbb{U}_L := \left(1/\overline{L}\right)^{\leftarrow}$ with

$$L^{\leftarrow}(u) := \inf\{v : L(v) \geq u\},\ \text{for } 0<u<1,$$

denoting the (left-continuous) the quantile function pertaining to a (right-continuous) df L. Since $\overline{F} \in \mathcal{RV}_{(-1/\gamma)}$ and $\overline{G} \in \mathcal{RV}_{(-1/\gamma_2)}$, then we may assume

$$\overline{F} \in 2\mathcal{RV}_{(-1/\gamma)}(A_F,\rho_F)\ \text{and}\ \overline{G} \in 2\mathcal{RV}_{(-1/\gamma_2)}(A_G,\rho_G). \tag{2.1}$$

Since $\overline{H} \in \mathcal{RV}_{(-1/\beta)}$, thus we may also suppose that

$$\overline{H} \in 2\mathcal{RV}_{(-1/\beta)}(A_H,\rho_H). \tag{2.2}$$

Theorem 2.1. *Assume that condition* (2.1) *holds. Let $k=k_n$ be an integer sequence satisfying $k\to\infty$ and $k/n\to 0$. In addition, if condition* (2.2) *is fulfilled, then, there exists a sequence of Wiener processes $\{W_n(x),\ x\geq 0\}_{n\geq 1}$, such that for every small $0<\nu<1$, we have*

$$\sup_{x\geq 1} x^{\nu}\left|D_n^{(i)}(x)\right| \overset{\mathbf{P}}{\to} 0,\ i=1,2. \tag{2.3}$$

Moreover

$$\sup_{x\geq 1} x^{\nu}\left|\sqrt{k}D_n^{(i)}(x) - \mathcal{L}_n^{(i)}(x) - \sqrt{k}\mathcal{B}_n^{(i)}(x)\right| \overset{\mathbf{P}}{\to} 0,\ i=1,2, \tag{2.4}$$

provided that $\sqrt{k}\mathbb{A}_F(n/k)$, $\sqrt{k}\mathbb{A}_G(n/k)$ and $\sqrt{k}\mathbb{A}_H(n/k)$ are asymptotically bounded, where

$$\mathcal{L}_n^{(1)}(x) := W_n\left(x^{-1/\gamma}\right) - x^{-1/\gamma}W_n(1)$$

and

$$\begin{aligned}(\beta/\gamma)\mathcal{L}_n^{(2)}(x) :=\ & x^{-1/\beta}\left\{x^{1/\gamma}W_n\left(x^{-1/\gamma}\right) - W_n(1)\right\} \\ & + (1-\gamma/\beta)\int_0^{x^{-1/\gamma}} s^{\gamma/\beta-2}W_n(s)\,ds \\ & - (1-\gamma/\beta)x^{-1/\beta}\int_0^1 s^{\gamma/\beta-2}W_n(s)\,ds,\end{aligned}$$

with

$$\mathcal{B}_n^{(1)}(x) := x^{-1/\gamma}\frac{x^{\rho_F/\gamma}-1}{\rho_F\gamma}\mathbb{A}_F(n/k)\ \text{and}\ \mathcal{B}_n^{(2)}(x) := x^{-1/\beta}\frac{x^{\rho_H/\beta}-1}{\rho_H\beta}\mathbb{A}_H(n/k).$$

Thereby, in view of the representation (1.4) and by using respectively the two results of Theorem 2.1 we end up with the consistency and asymptotic normality of $\widehat{\gamma}_1$, given in the following theorem.

Theorem 2.2. *Assume that* (2.1) *holds. Let $k=k_n$ be an integer sequence satisfying $k\to\infty$ and $k/n\to 0$, then*

$$\widehat{\gamma}_1 \overset{\mathbf{P}}{\to} \gamma_1,\ \text{as } N\to\infty.$$

In addition, if (2.2) *is fulfilled, then*

$$\sqrt{k}\left(\widehat{\gamma}_1 - \gamma_1\right) = Z_{n1} + Z_{n2} + \mu + o_{\mathbf{P}}(1),$$

where

$$\frac{(\gamma-2\beta)^2}{2\beta^2}Z_{n1} := \gamma\int_0^1 s^{-1}W_n(s)\,ds-\gamma W_n(1)$$

and

$$-\frac{(\gamma-2\beta)^2}{\gamma^2}Z_{n2} := (2\gamma-\beta)\frac{\gamma}{\beta}\int_0^1 s^{\gamma/\beta-2}W_n(s)\,ds-\gamma W_n(1)$$
$$+\left(\frac{\gamma}{\beta}-1\right)\frac{\gamma^2}{\beta}\int_0^1 s^{\gamma/\beta-2}W_n(s)(\log s)\,ds,$$

provided that $\sqrt{k}\mathbb{A}_G(n/k)=O(1)$, $\sqrt{k}\mathbb{A}_F(n/k)\to\lambda_F$ *and* $\sqrt{k}\mathbb{A}_H(n/k)\to\lambda_H$, *where*

$$\mu := \frac{2\beta^2(\gamma-2\beta)^{-2}\lambda_F}{1-\rho_F}-\frac{\gamma^2(\gamma-2\beta)^{-2}\lambda_H}{1-\rho_H}. \tag{2.5}$$

This implies that

$$\sqrt{k}\left(\widehat{\gamma}_1-\gamma_1\right)\to\mathcal{N}\left(\mu,\sigma^2\right),\ \textit{as } N\to\infty,$$

where

$$\sigma_2^2 := \frac{\gamma^6\beta\left(\beta^2-2\beta\gamma+2\gamma^2\right)}{(2\gamma-\beta)^3(\gamma-2\beta)^4}.$$

Remark 2.3. The complete data case corresponds to the situation when $\beta\equiv\gamma$, in which case we have $\gamma\equiv\gamma_1$. It follows that $\sqrt{k}\left(\widehat{\gamma}_1-\gamma_1\right)\overset{\mathcal{D}}{\to}\mathcal{N}\left(\lambda/(1-\rho_F),\gamma_1^2\right)$, as $N\to\infty$, which meets the asymptotic normality of the classical Hill estimator [10], see for instance, Theorem 3.2.5 in [9].

Remark 2.4. In terms of the tail indices γ_1 and γ_2, we have

$$\sigma_2^2=\gamma_2^3\frac{(\gamma_1/\gamma_2)^2\left(2(\gamma_1/\gamma_2)+1\right)^4\left(5(\gamma_1/\gamma_2)^2+4\gamma_1/\gamma_2+1\right)}{(\gamma_1/\gamma_2+1)\left(3(\gamma_1/\gamma_2)+1\right)^3}.$$

Remark 2.5. We show that the ratio between the asymptotic variances σ_1^2 and σ_2^2 equals

$$\frac{\sigma_1^2}{\sigma_2^2}=\frac{\left(1-x^3\right)(2x+1)^4\left(5x^2+4x+1\right)}{\left(1+x^2\right)(3x+1)^3},\ \text{where } x:=\gamma_1/\gamma_2,$$

and

$$\begin{cases} 1<\sigma_1^2/\sigma_2^2<3.2, & \text{for } 0<\gamma_1/\gamma_2<0.94125 \\ 0<\sigma_1^2/\sigma_2^2<1, & \text{for } 0.94125<\gamma_1/\gamma_2<1. \end{cases}$$

The curve of ratio σ_1^2/σ_2^2 in the interval $(0,1)$, given in Figure **??**, illustrates the previous inequalities. We conclude that $\widehat{\gamma}_1$ is asymptotically more efficient than $\widehat{\gamma}_1^{(\mathbf{BMN})}$ for $0<\gamma_1/\gamma_2<0.94125$, otherwise $\widehat{\gamma}_1^{(\mathbf{BMN})}$ is asymptotically more efficient than $\widehat{\gamma}_1$. It is worth mentioning that the comparison is made for $0<\gamma_1/\gamma_2<1$, because the asymptotic normality of $\widehat{\gamma}_1^{(\mathbf{BMN})}$ is established only for $0<\gamma_1<\gamma_2$.

3 Simulation study

In this section, we check the finite sample behavior of $\widehat{\gamma}_1$ compared with $\widehat{\gamma}_1^{(\mathbf{BMN})}$ and $\widehat{\gamma}_1^{(\mathbf{GS})}$ in terms of absolute bias and rmse. To this end, let us consider sets of truncated and truncation data drawn from Burr (γ,δ) and Fréchet (γ) models with respective df's

$$\overline{\mathcal{F}}(x)=\left(1+x^{1/\delta}\right)^{-\delta/\gamma},\ x\geq0,\ \delta>0,\ \gamma>0;$$

and

$$\overline{\mathcal{F}}(x)=1-\exp\left(-x^{-1/\gamma}\right),\ x\geq0,\ \gamma>0.$$

Let consider the following scenarios that correspond to df's F^* and G^*:

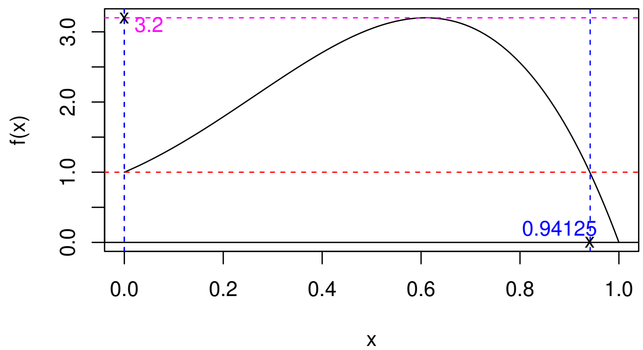

Fig. 2.1. Plotting of the ratio $f(x) := \sigma_1^2/\sigma_2^2$ as function of $x := \gamma_1/\gamma_2$ on the interval $(0,1)$.

	$\gamma_1 = 0.6$, $\delta = 1/4$		
	p		
	30% (90%)	40% (60%)	
	γ_2		
[S1]	0.257 (5.4)	0.093 (3.843)	[S2]
[S3]	0.167 (4.701)	0.420 (5.272)	[S4]

TABLE 1. Choices of the tail indices and corresponding percentages of observed sample for each scenario.

- [S1] Burr (γ_1, δ) truncated by Burr(γ_2, δ)
- [S2] Fréchet (γ_1) truncated by Fréchet (γ_2)
- [S3] Fréchet(γ_1) truncated by Burr(γ_2, δ)
- [S4] Burr(γ_1, δ) truncated by Fréchet(γ_2)

First, we fix the values 0.6 for γ_1 and $1/4$ for δ, then choose different values for γ_2 so that the percentage of observed data p given in (1.2), be around of 40% and 60% for both scenarios [S2] and [S4] while we choose 30% and 90% for both scenarios [S1] and [S3]. The choice of parameters provides couples of (γ_1, γ_2) of different order, that is $\gamma_1 < \gamma_2$ and $\gamma_1 > \gamma_2$, which may be obtained by numerically solve, in γ_2, Equation (1.2). The results are recapitulated in the following table:
Thereby, for each scenario, we choose two triplets of parameters (γ_1, γ_2, p) as follows:

- $S1 : (\gamma_1, \gamma_2, p) = (0.6, 0.257, 30\%) ; (0.6, 4.701.4, 90\%)$
- $S2 : (\gamma_1, \gamma_2, p) = (0.6, 0.093, 40\%) ; (0.6, 3.843, 60\%)$
- $S3 : (\gamma_1, \gamma_2, p) = (0.6, 0.167, 30\%) ; (0.6, 4.701, 90\%)$
- $S4 : (\gamma_1, \gamma_2, p) = (0.6, 0.420, 40\%) ; (0.6, 5.272, 60\%)$

We vary the common size $N = 300, 500, 1000, 1500$ of both samples $(\mathbf{X}_1, ..., \mathbf{X}_N)$ and $(\mathbf{Y}_1, ..., \mathbf{Y}_N)$, then for each size, we generate 1000 independent replicates. For the selection of the optimal numbers of upper order statistics used in the computation of the three aforementioned estimators, we apply the algorithm of [15] page 137. Our illustrations and comparison are made with respect to the absolute biases (abias) and rmse's, which are summarized in the four Tables 2-3-4-5 and the eight Figures 3.2-3.3-3.4-3.5-3.6-3.7-3.8-3.9. In the light of all tables and Figures , the overall conclusion is that $\widehat{\gamma}_1$ behaves well both in terms of bias and rmse and having a finite sample behavior almost close to $\widehat{\gamma}_1^{(\mathbf{BMN})}$. Moreover, both the two estimators perform better than $\widehat{\gamma}_1^{(\mathbf{GS})}$ in particular in small sample case and for small percentage of observed data p, on the other termes the later becomes unstable for small sample sizes. On the other hand, as noted in two Remarks 2.4 and 2.5, that $\widehat{\gamma}_1$ is asymptotically more efficient than $\widehat{\gamma}_1^{(\mathbf{BMN})}$ for "almost" all positive couples (γ_1, γ_2) , which also makes our new estimator more advantageous regarding to the two other ones.

		$\widehat{\gamma}_1$		$\widehat{\gamma}_1^{(\mathbf{BMN})}$		$\widehat{\gamma}_1^{(\mathbf{GS})}$	
				$p = 0.3$			
N	n	abias	rmse	abias	rmse	abias	rmse
300	90	0.398	0.448	0.403	0.442	0.445	4.911
500	149	0.236	0.469	0.226	0.320	0.418	3.989
1000	300	0.187	0.459	0.171	0.276	0.460	2.731
1500	450	0.144	0.362	0.144	0.276	0.342	1.830
				$p = 0.9$			
300	270	0.007	0.138	0.004	0.138	0.042	0.346
500	449	0.001	0.110	0.002	0.110	0.016	0.172
1000	899	0.008	0.076	0.005	0.076	0.021	0.117
1500	1350	0.004	0.065	0.002	0.065	0.019	0.098

TABLE 2. Absolute biases and rmse's for the tail index estimators correspond to scenario S1 based on 1000 right-truncated samples.

		$\widehat{\gamma}_1$		$\widehat{\gamma}_1^{(\mathbf{BMN})}$		$\widehat{\gamma}_1^{(\mathbf{GS})}$	
				$p = 0.4$			
N	n	abias	rmse	abias	rmse	abias	rmse
300	125	0.340	0.577	0.319	0.361	0.606	5.983
500	208	0.345	0.565	0.273	0.327	0.607	5.896
1000	416	0.264	0.509	0.243	0.298	0.428	1.326
1500	626	0.212	0.423	0.218	0.279	0.440	1.760
				$p = 0.6$			
300	182	0.008	0.163	0.012	0.164	0.054	7.398
500	304	0.010	0.127	0.014	0.127	0.001	0.208
1000	608	0.008	0.091	0.010	0.091	0.004	0.145
1500	912	0.009	0.076	0.011	0.077	0.004	0.124

TABLE 3. Absolute biases and rmse's for the tail index estimators correspond to scenario S2, based on 1000 right-truncated samples.

4 CONCLUDING NOTES

By using the well-known probability weighted moment estimation method, we derived a new estimator of the tail index for right truncated heavy-tailed data and established its consistency and asymptotic normality without additional assumptions on the underlying df's. Moreover, the proposed method may also serve to estimate the second order parameter ρ_F which is of practical relevance in extreme value

		$\widehat{\gamma}_1$		$\widehat{\gamma}_1^{(\mathbf{BMN})}$		$\widehat{\gamma}_1^{(\mathbf{GS})}$	
				$p = 0.3$			
N	n	abias	rmse	abias	rmse	abias	rmse
300	99	0.371	0.490	0.357	0.411	0.681	4.827
500	165	0.226	0.574	0.233	0.310	0.599	1.803
1000	331	0.181	0.465	0.165	0.278	0.357	2.912
1500	498	0.179	0.401	0.153	0.267	0.422	1.829
				$p = 0.9$			
300	273	0.023	0.147	0.027	0.147	0.513	11.574
500	456	0.011	0.110	0.014	0.110	0.024	0.158
1000	911	0.008	0.078	0.010	0.078	0.011	0.118
1500	1367	0.010	0.067	0.012	0.067	0.005	0.098

TABLE 4. Absolute biases and rmse's for the tail index estimators correspond to scenario S3 , based on 1000 right-truncated samples.

		$\widehat{\gamma}_1$		$\widehat{\gamma}_1^{(\mathbf{BMN})}$		$\widehat{\gamma}_1^{(\mathbf{GS})}$	
$p = 0.4$							
N	n	abias	rmse	abias	rmse	abias	rmse
300	121	0.177	0.494	0.170	0.317	0.975	11.564
500	201	0.080	0.594	0.112	0.297	0.288	3.733
1000	403	0.059	0.327	0.093	0.247	0.189	1.231
1500	604	0.047	0.256	0.062	0.246	0.164	3.483
$p = 0.6$							
300	180	0.011	0.158	0.007	0.156	0.057	5.154
500	300	0.008	0.126	0.005	0.125	0.170	4.699
1000	601	0.006	0.091	0.002	0.091	0.011	0.150
1500	902	0.006	0.076	0.003	0.076	0.002	0.116

TABLE 5. Absolute biases and rmse's for the tail index estimators correspond to scenario S4 , based on 1000 right-truncated samples.

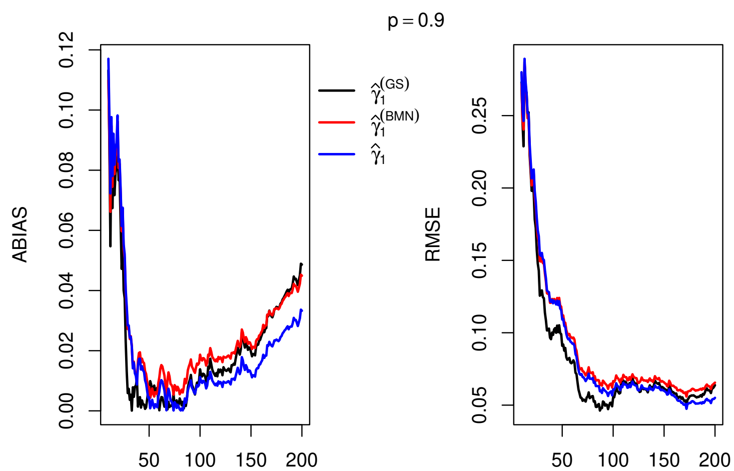

Fig. 3.2. Absolute bias (left panel) and RMSE (right panel) of $\widehat{\gamma}_1$ (blue) and $\widehat{\gamma}_1^{(BMN)}$ (red) and $\widehat{\gamma}_1^{(GS)}$(black), corresponding to scenario $S1$: ($\gamma_1 = 0.6$, $\gamma_2 = 5.4$ and $p = 90\%$) based on 1000 samples of size 500

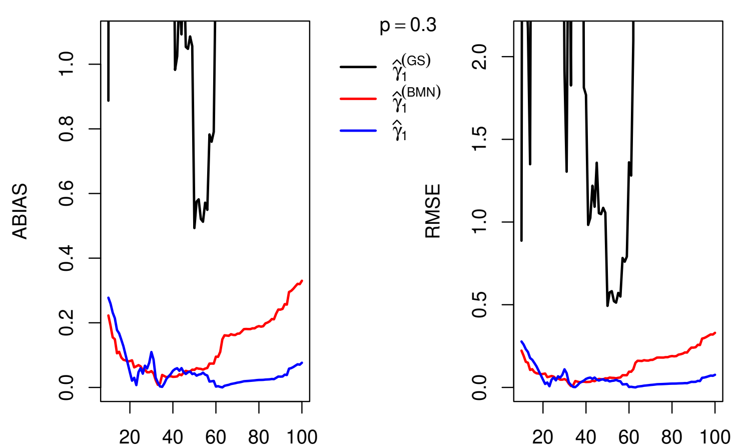

Fig. 3.3. Absolute bias (left panel) and RMSE (right panel) of $\widehat{\gamma}_1$ (blue) and $\widehat{\gamma}_1^{(BMN)}$ (red) and $\widehat{\gamma}_1^{(GS)}$(black), corresponding to scenario $S1$: ($\gamma_1 = 0.6$, $\gamma_2 = 0.257$ and $p = 30\%$) based on 1000 samples of size 500

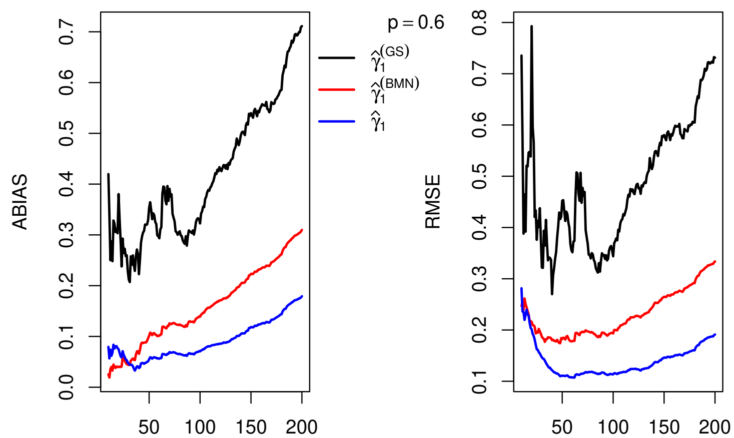

Fig. 3.4. Absolute bias (left panel) and RMSE (right panel) of $\widehat{\gamma}_1$ (blue) and $\widehat{\gamma}_1^{(BMN)}$ (red) and $\widehat{\gamma}_1^{(GS)}$(black), corresponding to scenario $S2$: ($\gamma_1 = 0.6$, $\gamma_2 = 3.843$ and $p = 60\%$) based on 1000 samples of size 500

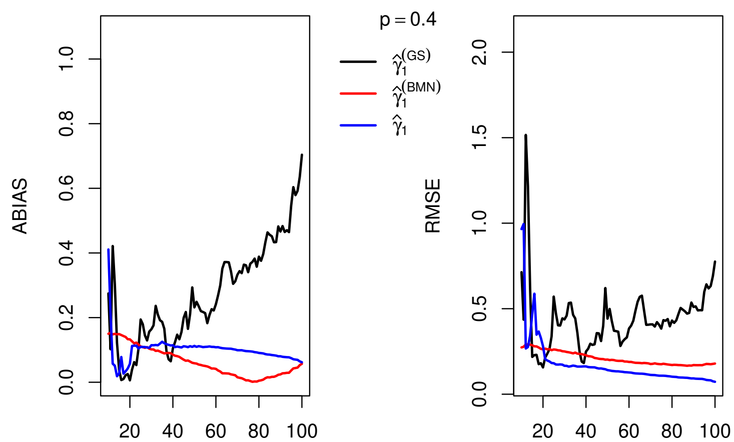

Fig. 3.5. Absolute bias (left panel) and RMSE (right panel) of $\widehat{\gamma}_1$ (blue) and $\widehat{\gamma}_1^{(BMN)}$ (red) and $\widehat{\gamma}_1^{(GS)}$(black), corresponding to scenario $S2$: ($\gamma_1 = 0.6$, $\gamma_2 = 0.093$ and $p = 40\%$) based on 1000 samples of size 500

analysis due its crucial importance in selecting the optimal number of upper order statistics k in tail index estimation (see, e.g., [9]) and to reduce the bias of such estimation. The asymptotic behavior of the obtained reduced bias estimator may be also established by means of the two tail empirical processes $D_n^{(i)}(x)$, $i = 1, 2$. This problem will be addressed in our future work.

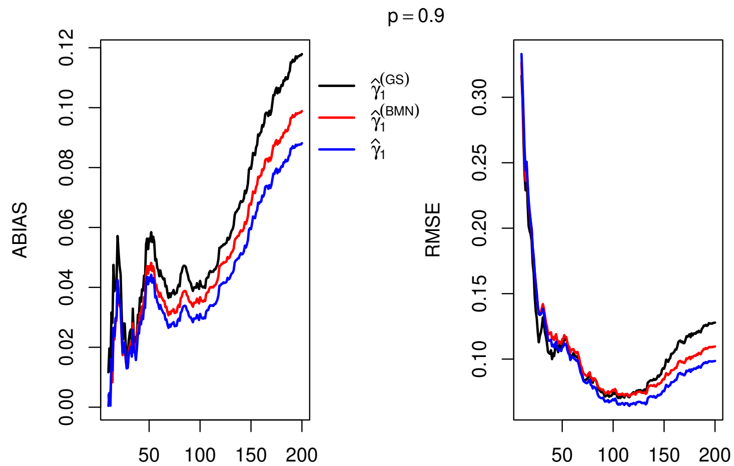

Fig. 3.6. Absolute bias (left panel) and RMSE (right panel) of $\widehat{\gamma}_1$ (blue) and $\widehat{\gamma}_1^{(BMN)}$ (red) and $\widehat{\gamma}_1^{(GS)}$(black), corresponding to scenario $S3$: ($\gamma_1 = 0.6$, $\gamma_2 = 4.701$ and $p = 90\%$) based on 1000 samples of size 500

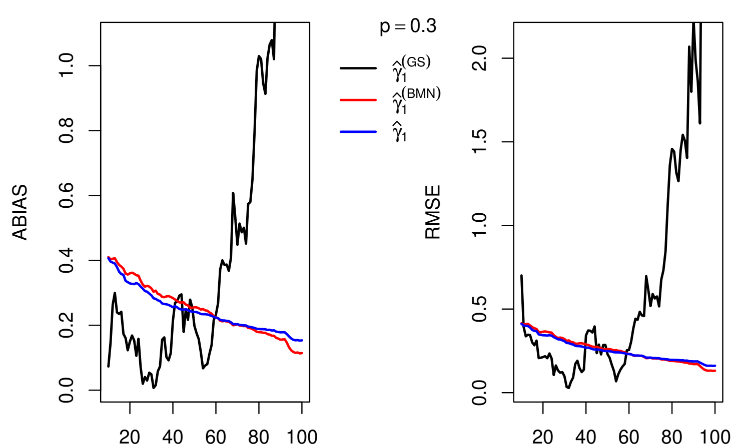

Fig. 3.7. Absolute bias (left panel) and RMSE (right panel) of $\widehat{\gamma}_1$ (blue) and $\widehat{\gamma}_1^{(BMN)}$ (red) and $\widehat{\gamma}_1^{(GS)}$(black), corresponding to scenario $S3$: ($\gamma_1 = 0.6$, $\gamma_2 = 0.167$ and $p = 30\%$) based on 1000 samples of size 500

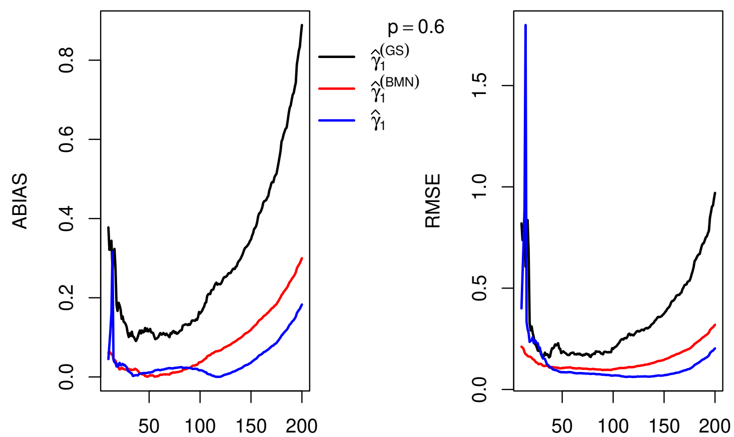

Fig. 3.8. Absolute bias (left panel) and RMSE (right panel) of $\widehat{\gamma}_1$ (blue) and $\widehat{\gamma}_1^{(BMN)}$ (red) and $\widehat{\gamma}_1^{(GS)}$(black), corresponding to scenario $S4$: ($\gamma_1 = 0.6$, $\gamma_2 = 5.272$ and $p = 60\%$) based on 1000 samples of size 500

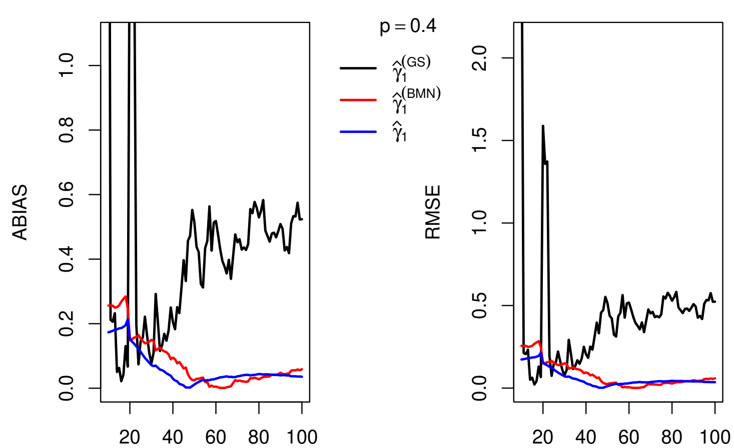

Fig. 3.9. Absolute bias (left panel) and RMSE (right panel) of $\widehat{\gamma}_1$ (blue) and $\widehat{\gamma}_1^{(BMN)}$ (red) and $\widehat{\gamma}_1^{(GS)}$(black), corresponding to scenario $S4$: ($\gamma_1 = 0.6$, $\gamma_2 = 0.420$ and $p = 40\%$) based on 1000 samples of size 500

Acknowledgments

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5 APPENDIX A

It is worth mentioning that, since $n/N \to p$ a.s. as $N \to \infty$, then for a random sequence $Z_N \overset{\mathbf{P}}{\to} Z$ as $N \to \infty$, we have $Z_n \overset{\mathbf{P}}{\to} Z$ as $N \to \infty$, too. The proof of this matter is similar as that is used in Lemma 3.7 in [3]. In other words, the results regarding to convergence in probability of a sequence of rv's indexed by N can also be used by indexing by n.

5.1 Proof of Theorem 2.1

We will only show the results of the Theorem for $i = 2$, since those of case $i = 1$ become trivial when replacing β by γ. To start, first recall that $D_n^{(2)}(x) = M_n^{(2)}(x) - x^{-1/\beta}$, $x \geq 1$, where

$$M_n^{(2)}(x) = \frac{\overline{H}_n(xX_{n-k:n})}{\overline{H}_n(X_{n-k:n})},$$

and

$$\overline{H}_n(x) = \frac{\int_x^\infty \overline{G}_n(w)\, dF_n(w)}{\int_0^\infty \overline{G}_n(w)\, dF_n(w)}.$$

Observe that $M_n^{(2)}(x) = \Delta_n^{(2)}(x) / \Delta_n^{(2)}(1)$, where

$$\Delta_n^{(2)}(x) := \frac{\frac{n}{k}\int_{xX_{n-k:n}}^\infty \overline{G}_n(w)\, dF_n(w)}{\overline{G}_n(X_{n-k:n})}.$$

Thus, we may write

$$D_n^{(2)}(x) = \frac{\Delta_n^{(2)}(x) - (\beta/\gamma)\,x^{-1/\beta}}{\Delta_n^{(2)}(1)} - x^{-1/\beta}\frac{\Delta_n^{(2)}(1) - \beta/\gamma}{\Delta_n^{(2)}(1)}. \tag{5.1}$$

Let $a_k := \mathbb{U}_F(n/k)$ and decompose $\Delta_n^{(2)}(x) - (\beta/\gamma)\,x^{-1/\beta}$ into the sum of

$$T_{n1}(x) := \frac{n}{k}\left(\frac{1}{\overline{G}_n(X_{n-k:n})} - \frac{1}{\overline{G}(X_{n-k:n})}\right)\int_{xX_{n-k:n}}^{\infty}\overline{G}_n(w)\,dF_n(w),$$

$$T_{n2}(x) := \frac{n/k}{\overline{G}(X_{n-k:n})}\int_{xX_{n-k:n}}^{\infty}\left(\overline{G}_n(w) - \overline{G}(w)\right)dF_n(w),$$

$$T_{n3}(x) := \frac{n}{k}\left(\frac{1}{\overline{G}(X_{n-k:n})} - \frac{1}{\overline{G}(a_k)}\right)\int_{xX_{n-k:n}}^{\infty}\overline{G}(w)\,dF_n(w),$$

$$T_{n4}(x) := \frac{n/k}{\overline{G}(a_k)}\int_{xX_{n-k:n}}^{xa_k}\overline{G}(w)\,dF_n(w),$$

$$T_{n5}(x) := \frac{n/k}{\overline{G}(a_k)}\int_{xa_k}^{\infty}\overline{G}(w)\,d\left(F_n(w) - F(w)\right)$$

and

$$T_{n6}(x) := \frac{n/k}{\overline{G}(a_k)}\int_{xa_k}^{\infty}\overline{G}(w)\,dF(w) - (\beta/\gamma)\,x^{-1/\beta}.$$

Making use of (5.1), we way write

$$\Delta_n^{(2)}(1)\,D_n^{(2)}(x) = \sum_{i=1}^{5}\left(T_{ni}(x) - x^{-1/\beta}T_{ni}(1)\right) + \widetilde{\mathcal{B}}_n^{(2)}(x), \tag{5.2}$$

where

$$\begin{aligned}\widetilde{\mathcal{B}}_n^{(2)}(x) &:= T_{n6}(x) - x^{-1/\beta}T_{n6}(1)\\ &= \frac{n/k}{\overline{G}(a_k)}\int_{xa_k}^{\infty}\overline{G}(w)\,dF(w) - x^{-1/\beta}\frac{n/k}{\overline{G}(a_k)}\int_{a_k}^{\infty}\overline{G}(w)\,dF(w).\end{aligned}$$

Next, we show that for every sufficiently small $0 < \eta, \epsilon < 1/2$, we have

$$\sqrt{k}T_{ni}(x) = o_{\mathbf{P}}\left(\varrho(x)\right),\ \text{for } i = 1, 2, \tag{5.3}$$

$$\sqrt{k}\left(T_{n3}(x) + T_{n4}(x) + T_{n5}(x)\right) = \mathcal{L}_n^{(2)}(x) + o_{\mathbf{P}}\left(\varrho(x)\right) \tag{5.4}$$

and

$$\widetilde{\mathcal{B}}_n^{(2)}(x) = x^{-1/\beta}\left(\frac{x^{\rho_H/\beta} - 1}{\rho_H\gamma} + o\left(x^{\epsilon}\right)\right)\mathbb{A}_H(n/k), \tag{5.5}$$

uniformly over $x \geq 1$, where $\varrho(x) = x^{-\eta/\beta+\epsilon}$ and $\mathcal{L}^{(2)}(x)$ is the Gaussian process given in Theorem 2.1.

5.1.1 Preliminaries

Note that F^* and G^* are continuous, then it is easy to verify that both df's F and G are as well, therefore the two rv's $U := \overline{F}(X)$ and $V := \overline{G}(Y)$ are uniformly distributed on $(0,1)$. Let

$$\mathcal{U}_n(s) := n^{-1}\sum_{i=1}^{n}\mathbb{I}_{\{U_i\leq s\}} \text{ and } \mathcal{V}_n(s) := n^{-1}\sum_{i=1}^{n}\mathbb{I}_{\{V_i\leq s\}},$$

denote the uniform empirical df's pertaining to the samples

$$U_i := \overline{F}(X_i) \text{ and } V_i := \overline{G}(Y_i),\ i=1,...,n,$$

respectively. We have $\overline{F}(x) = \overline{F}(x+)$, then

$$\mathbb{I}\left(U_i \leq \overline{F}(x)\right) = \mathbb{I}\left(\overline{F}(X_i) \leq \overline{F}(x+)\right)$$

and since $\overline{F}$ is decreasing then this latter equals

$$\mathbb{I}(X_i \geq x+) = 1 - \mathbb{I}(X_i < x+) = 1 - \mathbb{I}(X_i \leq x).$$

By using similar arguments, we end up with

$$\mathbb{I}(Y_i \geq y+) = 1 - \mathbb{I}(Y_i < y+) = 1 - \mathbb{I}(Y_i \leq y).$$

Hence for $x, y \geq 0$, we may write

$$\overline{F}_n(y) = \mathcal{U}_n\left(\overline{F}(x)\right) \text{ and } \overline{G}_n(y) = \mathcal{V}_n\left(\overline{G}(y)\right). \tag{5.6}$$

Next, we will use a useful weak approximation, due to [6], corresponding to the uniform tail empirical processes, saying that: in the probability space $(\Omega, \mathcal{A}, \mathbf{P})$, there exists a sequence of standard Wiener precesses $\{W_n(x),\ x\geq 0\}$, such that, for every $0<\eta<1/2$ and $M>0$, we have

$$\sup_{0<s\leq M} s^{-\eta}\left|\sqrt{k}\left(\frac{n}{k}\mathcal{U}_n\left(\frac{k}{n}s\right)-s\right)-W_n(s)\right| = o_{\mathbf{P}}(1). \tag{5.7}$$

On the other hand, we have $\sup_{0<s\leq M} s^{-\eta}|W_n(s)| = O_{\mathbf{P}}(1)$ [**?**, see, e.g., example 1.8 in]]Alex86, which implies that

$$\sup_{0<s\leq M} s^{-\eta}\left|\sqrt{k}\left(\frac{n}{k}\mathcal{U}_n\left(\frac{k}{n}s\right)-s\right)\right| = O_{\mathbf{P}}(1). \tag{5.8}$$

The previous result remains valid when replacing $\mathcal{U}_n$ by $\mathcal{V}_n$, that is

$$\sup_{0<s\leq M} s^{-\eta}\left|\sqrt{k}\left(\frac{n}{k}\mathcal{V}_n\left(\frac{k}{n}s\right)-s\right)\right| = O_{\mathbf{P}}(1). \tag{5.9}$$

5.1.2 Asymptotic behavior of T_{n1}

Note that $\overline{F}(a_k) = k/n$, and let us write

$$\begin{aligned}\sqrt{k}T_{n1} &= -\frac{\sqrt{k}\left(\overline{G}_n(X_{n-k:n}) - \overline{G}(X_{n-k:n})\right)}{\overline{G}_n(X_{n-k:n})} \\ &\quad\times\int_x^\infty \frac{\overline{G}_n(wX_{n-k:n})}{\overline{G}(X_{n-k:n})}d\frac{F_n(wX_{n-k:n})}{\overline{F}(a_k)}.\end{aligned}$$

Observe that, by letting $s = \dfrac{n}{k}\overline{G}(X_{n-k:n})$, we have

$$\sqrt{k}\left(\overline{G}_n(X_{n-k:n}) - \overline{G}(X_{n-k:n})\right) = \frac{k}{n}\sqrt{k}\left(\frac{n}{k}\mathcal{V}_n\left(\frac{k}{n}s\right)-s\right),$$

which, by using the result (5.9), equals

$$O_{\mathbf{P}}(1)(k/n)s^{\eta} = O_{\mathbf{P}}(1)(k/n)^{1-\eta}\left(\overline{G}(X_{n-k:n})\right)^{\eta},$$

for some fixed $0<\eta<1/2$. It is worth mentioning that, since $X_i<Y_i$, for $i=1,...,n$, then $X_{n:n}<Y_{n:n}$, which implies that $V_{1:n}=\overline{G}\left(Y_{n:n}\right)<\overline{G}\left(X_{n-k:n}\right)<1$, therefore

$$\frac{\overline{G}\left(X_{n-k:n}\right)}{\overline{G}_n\left(X_{n-k:n}\right)}=\frac{\overline{G}\left(X_{n-k:n}\right)}{\mathcal{V}_n\left(\overline{G}\left(X_{n-k:n}\right)\right)}<\sup_{V_{1:n}\leq s<1}\frac{s}{\mathcal{V}_n\left(s\right)},$$

which, from Proposition 6.2, is equal to $O_{\mathbf{P}}\left(1\right).$ On the other hand, $\overline{G}_n\left(w\right)=0$, for $w\geq Y_{n:n}$, it follows that

$$\sqrt{k}T_{n1}=O_{\mathbf{P}}\left(1\right)\left(\frac{k/n}{\overline{G}\left(X_{n-k:n}\right)}\right)^{1-\eta}\int_x^{Y_{n:n}/X_{n-k:n}}\frac{\overline{G}_n\left(wX_{n-k:n}\right)}{\overline{G}\left(X_{n-k:n}\right)}d\frac{F_n\left(wX_{n-k:n}\right)}{\overline{F}\left(a_k\right)}.$$

Observe now, that for any $w\geq 1$, we have

$$\frac{\overline{G}_n\left(wX_{n-k:n}\right)}{\overline{G}\left(X_{n-k:n}\right)}=\frac{\mathcal{V}_n\left(\overline{G}\left(wX_{n-k:n}\right)\right)}{\overline{G}\left(wX_{n-k:n}\right)},$$

and, since $\overline{G}\left(wX_{n-k:n}\right)\leq\overline{G}\left(X_{n-k:n}\right)$, then in view of Proposition 6.2 the latter ratio is less than or equal to $\sup_{V_{1:n}\leq s<1}\mathcal{V}_n\left(s\right)/s=O_{\mathbf{P}}\left(1\right)$, hence

$$\sqrt{k}T_{n1}=O_{\mathbf{P}}\left(1\right)\left(\frac{k/n}{\overline{G}\left(X_{n-k:n}\right)}\right)^{1-\eta}\int_x^{\infty}\frac{\overline{G}\left(wX_{n-k:n}\right)}{\overline{G}\left(X_{n-k:n}\right)}d\frac{F_n\left(wX_{n-k:n}\right)}{\overline{F}\left(a_k\right)}.$$

Next, we require to the following Potter-type inequalities [**?**, see, e.g., Proposition B.1.10, page 369 in]]deHF06 corresponding to regular variation functions.

Proposition 5.1. *Let $g\in\mathcal{RV}_{(\alpha)}$ with $\alpha\in\mathbb{R}$. Then, for any sufficiently small $\epsilon>0$, there exists $t_0=t_0\left(\epsilon\right)>0$, such that $\left|g\left(ts\right)/g\left(t\right)-s^{-\alpha}\right|\leq\epsilon s^{-\alpha}\max\left(s^{-\epsilon},s^{\epsilon}\right)$, for any $t\geq t_0$ and $s>0$.*

For the sake of simplicity, we set $x^{\nu\pm\epsilon}:=x^{\nu}\max\left(x^{-\epsilon},x^{\epsilon}\right)$ and $\pm\epsilon c=\pm\epsilon$ for $\epsilon\downarrow 0$ and any real constants ν and c. Note that $X_{n-k:n}/a_k\overset{\mathbf{P}}{\to}1$, then by using the previous proposition, we readily show that $\overline{G}\left(X_{n-k:n}\right)/\overline{G}\left(a_k\right)=O_{\mathbf{P}}\left(1\right)$ and

$$\frac{\overline{G}\left(wX_{n-k:n}\right)}{\overline{G}\left(X_{n-k:n}\right)}=O_{\mathbf{P}}\left(w^{-1/\gamma_2+\epsilon}\right),\text{ as }N\to\infty,$$

uniformly on $w\geq 1$. Thus

$$\sqrt{k}T_{n1}=O_{\mathbf{P}}\left(1\right)\left(\frac{k/n}{\overline{G}\left(a_k\right)}\right)^{1-\eta}\int_x^{\infty}w^{-1/\gamma_2+\epsilon}d\frac{\overline{F}_n\left(wX_{n-k:n}\right)}{\overline{F}\left(a_k\right)}.$$

Recall that , since $\overline{F}_n\left(w\right)=0$, for $w\geq X_{n:n}$ and $\overline{F}_n\left(wX_{n-k:n}\right)=\mathcal{U}_n\left(\overline{F}\left(wX_{n-k:n}\right)\right)$, then by using an integration by parts to the latter integral and then Proposition 6.2, yields

$$\begin{aligned}\sqrt{k}T_{n1} &= O_{\mathbf{P}}\left(1\right)\left(\frac{k/n}{\overline{G}\left(a_k\right)}\right)^{1-\eta}\\ &\times\left\{x^{-1/\gamma_2+\epsilon}\frac{\overline{F}\left(xX_{n-k:n}\right)}{\overline{F}\left(a_k\right)}+\int_x^{\infty}\frac{\overline{F}\left(wX_{n-k:n}\right)}{\overline{F}\left(a_k\right)}dw^{-1/\gamma_2+\epsilon}\right\}.\end{aligned}$$

Making use of Proposition 5.1 and after integration, we show that both two quantities between brackets equal $O_{\mathbf{P}}\left(x^{-1/\gamma_1-1/\gamma+\epsilon}\right)=O_{\mathbf{P}}\left(\varrho\left(x\right)\right)$ uniformly on $x\geq 1$. On the hand in view of Proposition 6.4, we have $\overline{G}\left(a_k\right)=O\left(1\right)\left(k/n\right)^{\gamma/\gamma_2}$, then

$$\left(\frac{k/n}{\overline{G}\left(a_k\right)}\right)^{1-\eta}=\left(k/n\right)^{\left(1-\eta\right)\left(1-\gamma/\gamma_2\right)}.$$

Recall that $0<\eta<1/2$ and $0<\gamma/\gamma_2<1$, then $\left(k/n\right)^{\left(1-\eta\right)\left(1-\gamma/\gamma_2\right)}=o\left(1\right)$, it follows that $\sqrt{k}T_{n1}=o_{\mathbf{P}}\left(\varrho\left(x\right)\right)$, uniformly on $x\geq 1$.

5.1.3 Asymptotic behavior of T_{n2}

It is clear that

$$\sqrt{k}T_{n2} = \frac{1}{\overline{G}\left(X_{n-k:n}\right)}\int_x^\infty \sqrt{k}\left(\overline{G}_n\left(wX_{n-k:n}\right)-\overline{G}\left(wX_{n-k:n}\right)\right)d\frac{F_n\left(wX_{n-k:n}\right)}{\overline{F}\left(a_k\right)}.$$

For convenience, we set $b_k := \mathbb{U}_G\left(n/k\right)$ so that $\overline{G}\left(b_k\right)=k/n$. It is easy to verify, from (5.6), that

$$\begin{aligned}&\frac{n}{k}\sqrt{k}\left(\overline{G}_n\left(wX_{n-k:n}\right)-\overline{G}\left(wX_{n-k:n}\right)\right)\\ =\ &\sqrt{k}\left(\frac{n}{k}\mathcal{V}_n\left(\frac{k}{n}\overline{G}\left(wX_{n-k:n}\right)/\overline{G}\left(b_k\right)\right)-\overline{G}\left(wX_{n-k:n}\right)/\overline{G}\left(b_k\right)\right),\end{aligned}$$

which, by using (5.9), equals $O_{\mathbf{P}}\left(1\right)\left(\overline{G}\left(wX_{n-k:n}\right)/b_k\right)^\eta$, uniformly on $w\geq 1$, therefore

$$\sqrt{k}T_{n2} = O_{\mathbf{P}}\left(1\right)\frac{k/n}{\overline{G}\left(X_{n-k:n}\right)}\int_x^\infty\left(\overline{G}\left(wX_{n-k:n}\right)/\overline{G}\left(b_k\right)\right)^\eta d\frac{F_n\left(wX_{n-k:n}\right)}{\overline{F}\left(a_k\right)}.$$

By using the routine manipulations of two Propositions 6.2 and 5.1, we get

$$\sqrt{k}T_{n2} = O_{\mathbf{P}}\left(1\right)\frac{k/n}{\overline{G}\left(X_{n-k:n}\right)}\left(\frac{X_{n-k:n}}{b_k}\right)^{-\eta/\gamma_2\pm\epsilon}\int_x^\infty w^{-\eta/\gamma_2+\epsilon}dw^{-1/\gamma+\epsilon}.$$

Recall that $X_{n-k:n}=\left(1+o_{\mathbf{P}}\left(1\right)\right)a_k$, then by making use of Proposition 6.4, it is easy to verify that

$$\frac{k/n}{\overline{G}\left(X_{n-k:n}\right)}\left(\frac{X_{n-k:n}}{b_k}\right)^{-\eta/\gamma_2\pm\epsilon} = O_{\mathbf{P}}\left(1\right)\left(k/n\right)^{(1-\eta)(1-\gamma/\gamma_2)\pm\epsilon}.$$

Since $\gamma/\gamma_2<1$ and $\int_x^\infty w^{-\eta/\gamma_2+\epsilon}dw^{-1/\gamma+\epsilon}=O_{\mathbf{P}}\left(\varrho\left(x\right)\right)$, then $\sqrt{k}T_{n2}=o_{\mathbf{P}}\left(\varrho\left(x\right)\right)$, uniformly on $x\geq 1$.

5.1.4 Asymptotic behavior of T_{n3}

Observe now

$$\sqrt{k}T_{n3} = \sqrt{k}\left(\frac{\overline{G}\left(a_k\right)}{\overline{G}\left(X_{n-k:n}\right)}-1\right)\int_{xX_{n-k:n}}^\infty\frac{\overline{G}\left(w\right)}{\overline{G}\left(a_k\right)}d\frac{F_n\left(w\right)}{\overline{F}\left(a_k\right)},$$

and

$$\begin{aligned}&\sqrt{k}\left(\frac{\overline{G}\left(a_k\right)}{\overline{G}\left(X_{n-k:n}\right)}-1\right)\\ &=\sqrt{k}\left(\frac{\overline{G}\left(a_k\right)}{\overline{G}\left(X_{n-k:n}\right)}-\left(\frac{a_k}{X_{n-k:n}}\right)^{-1/\gamma_2}\right)+\sqrt{k}\left(\left(\frac{a_k}{X_{n-k:n}}\right)^{-1/\gamma_2}-1\right)\\ &=: I_{n1}+I_{n2}.\end{aligned}$$

Next we show that $I_{n1}=o_{\mathbf{P}}\left(1\right)$. Indeed, we have $\overline{G}\in 2\mathcal{RV}_{(-1/\gamma_2)}\left(A_G,\rho_G\right)$ which implies that for possibly different functions $\widetilde{A}_G$, with $\widetilde{A}_G\left(t\right)\sim A_G\left(t\right)$, as $t\rightarrow\infty$, and for each $0<\epsilon<1$, there exists $t_0=t_0\left(\epsilon\right)$, such that for all $tz\geq t_0$ we have

$$\left|\frac{\overline{G}\left(tz\right)/\overline{G}\left(t\right)-z^{-1/\gamma_2}}{\widetilde{A}_G\left(t\right)}-z^{-1/\gamma_2}\frac{z^{\rho_G/\gamma_2}-1}{\rho_G\gamma_2}\right|\leq\epsilon z^{-1/\gamma_2\pm\epsilon}. \tag{5.10}$$

[?, see, e.g., Proposition 4 and Remark 1 in]]HJ-2011. We will use this inequality with $t=t_n=X_{n-k:n}$ and $z=z_n=a_k/X_{n-k:n}$. Since $z_n=\left(1+o_{\mathbf{P}}\left(1\right)\right)$ then $z_n^{\rho_G/\gamma_2}-1=o_{\mathbf{P}}\left(1\right)$, it follows that

$$\frac{\overline{G}\left(a_k\right)}{\overline{G}\left(X_{n-k:n}\right)}-\left(\frac{a_k}{X_{n-k:n}}\right)^{-1/\gamma_2} = o_{\mathbf{P}}\left(1\right)\left|A_G\right|\left(X_{n-k:n}\right).$$

Since $|A_G|$ is regularly varying, then $A_G(X_{n-k:n}) = (1+o_{\mathbf{P}}(1)) A_G(a_k)$. Recall that by assumption $\sqrt{k}A_G(a_k) = \sqrt{k}\mathbb{A}_G(n/k) = O(1)$, it follows that $I_{n1} = o_{\mathbf{P}}(1)$. The term I_{n2} may be decomposed into

$$\sqrt{k}\left(\left(\frac{X_{n-k:n}}{a_k}\right)^{1/\gamma_2} - \left(\frac{nU_{k:n}}{k}\right)^{-\gamma/\gamma_2}\right) + \sqrt{k}\left(\left(\frac{nU_{k:n}}{k}\right)^{-\gamma/\gamma_2} - 1\right).$$

By using similar arguments as used for I_{n1}, we also show that the first of the previous quantity equals $o_{\mathbf{P}}(1)$. For the second term, we use assertion (i) in Proposition 6.2, to get

$$\sqrt{k}\left(\left(\frac{nU_{k:n}}{k}\right)^{-\gamma/\gamma_2} - 1\right) = \frac{\gamma}{\gamma_2}W_n(1) + o_{\mathbf{P}}(1).$$

It is now easy to show that

$$\int_{xX_{n-k:n}}^{\infty} \frac{\overline{G}(w)}{\overline{G}(a_k)} d\frac{F_n(w)}{\overline{F}(a_k)} = \frac{x^{-1/\gamma_2 - 1/\gamma}}{\gamma/\gamma_2 + 1} + o_{\mathbf{P}}(\varrho(x)),$$

therefore

$$\sqrt{k}T_{n3} = \frac{\gamma}{\gamma+\gamma_2}x^{-1/\gamma_2-1/\gamma}W_n(1) + o_{\mathbf{P}}(\varrho(x)),$$

uniformly on $x \geq 1$, which may be rewritten in terms of tail index $\beta = \gamma_1\gamma/(2\gamma_1 - \gamma)$, into

$$\sqrt{k}T_{n3} = (1-\beta/\gamma)\,x^{-1/\beta}W_n(1) + o_{\mathbf{P}}(\varrho(x)). \tag{5.11}$$

5.1.5 Asymptotic behavior of T_{n4}

Let us write

$$\sqrt{k}T_{n4}(x) = -\sqrt{k}\int_{xX_{n-k:n}/a_k}^{x} \frac{\overline{G}(wa_k)}{\overline{G}(a_k)} d\frac{\overline{F}_n(wa_k)}{\overline{F}(a_k)},$$

which may be decomposed into the sum of

$$\sqrt{k}T_{n4}^{(1)}(x) := -\sqrt{k}\int_{xX_{n-k:n}/a_k}^{x} \left(\frac{\overline{G}(wa_k)}{\overline{G}(a_k)} - w^{-1/\gamma_2}\right) d\frac{\overline{F}_n(wa_k)}{\overline{F}(a_k)},$$

$$\sqrt{k}T_{n4}^{(2)}(x) := -\sqrt{k}\int_{xX_{n-k:n}/a_k}^{x} w^{-1/\gamma_2} d\left(\frac{\overline{F}_n(wa_k) - \overline{F}(wa_k)}{\overline{F}(a_k)}\right),$$

$$\sqrt{k}T_{n4}^{(3)}(x) := -\sqrt{k}\int_{xX_{n-k:n}/a_k}^{x} w^{-1/\gamma_2} d\left(\frac{\overline{F}(wa_k)}{\overline{F}(a_k)} - w^{-1/\gamma}\right)$$

and

$$\sqrt{k}T_{n4}^{(4)}(x) := -\sqrt{k}\int_{xX_{n-k:n}/a_k}^{x} w^{-1/\gamma_2} dw^{-1/\gamma}.$$

We will show that $\sqrt{k}T_{n4}^{(1)}(x) = o_{\mathbf{P}}(\varrho(x))$, the proof of the other terms follow by using similar arguments. For convenience, we set $c_k^- := \min(1, X_{n-k:n}/a_k)$ and $c_k^+ := \min(1, X_{n-k:n}/a_k)$, and apply Proposition 5.1 (to $\overline{G}$), we get

$$\sqrt{k}T_{n4}^{(1)}(x) = o_{\mathbf{P}}(1)\sqrt{k}\int_{xc_k^-}^{xc_k^+} w^{-1/\gamma_2+\epsilon} d\frac{F_n(wa_k)}{\overline{F}(a_k)}.$$

Since $w^{-1/\gamma_2+\epsilon} < \left(xc_k^+\right)^{-1/\gamma_2+\epsilon}$ and $c_k^+ = 1 + o_{\mathbf{P}}(1)$, then

$$\sqrt{k}T_{n4}^{(1)}(x) = o_{\mathbf{P}}\left(x^{-1/\gamma_2+\epsilon}\right)\frac{\sqrt{k}\left|\overline{F}_n(xX_{n-k:n}) - \overline{F}_n(xa_k)\right|}{\overline{F}(a_k)}.$$

Observe that the previous ratio is less than or equal to

$$\sqrt{k}\frac{\left|\overline{F}_n(xX_{n-k:n}) - \overline{F}(xX_{n-k:n})\right|}{\overline{F}(a_k)} + \sqrt{k}\frac{\left|\overline{F}_n(xa_k) - \overline{F}_n(xa_k)\right|}{\overline{F}(a_k)}.$$

By applying (5.8) twice, we show that both terms equal $O_{\mathbf{P}}\left((k/n)^{1/2-\eta}\,x^{-\eta/\gamma+\epsilon}\right)$, it follows that

$$\sqrt{k}T_{n4}^{(1)}(x)=o_{\mathbf{P}}\left(x^{-1/\gamma_2-\eta/\gamma+\epsilon}\right)=o_{\mathbf{P}}\left(\varrho\left(x\right)\right).$$

For the last term we use an elementary integration to write

$$\sqrt{k}T_{n4}^{(4)}(x)=\frac{\gamma_2}{\gamma+\gamma_2}x^{-1/\gamma_2-1/\gamma}\sqrt{k}\left(\left(\frac{X_{n-k:n}}{a_k}\right)^{-1/\gamma_2-1/\gamma}-1\right),$$

then we make use of assertion (ii) in Proposition 6.4, we obtain

$$\sqrt{k}T_{n4}^{(4)}(x)=-x^{-1/\beta}W_n\left(1\right)+o_{\mathbf{P}}\left(\varrho\left(x\right)\right)=\sqrt{k}T_{n4}(x). \tag{5.12}$$

5.1.6 Asymptotic behavior of T_{n5}

Recall that

$$T_{n5}\left(x\right):=\frac{n/k}{\overline{G}\left(a_k\right)}\int_{xa_k}^{\infty}\overline{G}\left(w\right)d\left(F_n\left(w\right)-F\left(w\right)\right)$$

The change of variables $s=\overline{G}\left(w\right)/\overline{G}\left(a_k\right)$ gives $w=\mathbb{U}_G\left(1/\left(s\overline{G}\left(a_k\right)\right)\right)$ and therefore

$$T_{n5}=\frac{n}{k}\int_0^{\overline{G}(xa_k)/\overline{G}(a_k)}sd\left(\overline{F}_n\left(\mathbb{U}_G\left(\frac{1}{s\overline{G}\left(a_k\right)}\right)\right)-\overline{F}\left(\mathbb{U}_G\left(\frac{1}{s\overline{G}\left(a_k\right)}\right)\right)\right),$$

which by an integration by parts may be rewritten into the sum of

$$T_{n5}^{(1)}:=\frac{n}{k}\frac{\overline{G}\left(xa_k\right)}{\overline{G}\left(a_k\right)}\left(\overline{F}_n\left(xa_k\right)-\overline{F}\left(xa_k\right)\right)$$

and

$$T_{n5}^{(2)}:=-\frac{n}{k}\int_0^{\overline{G}(xa_k)/\overline{G}(a_k)}\left(\overline{F}_n\left(\mathbb{U}_G\left(\frac{1}{s\overline{G}\left(a_k\right)}\right)\right)-\overline{F}\left(\mathbb{U}_G\left(\frac{1}{s\overline{G}\left(a_k\right)}\right)\right)\right)ds.$$

Observe that

$$\sqrt{k}T_{n5}^{(1)}=\frac{\overline{G}\left(xa_k\right)}{\overline{G}\left(a_k\right)}\sqrt{k}\left\{\frac{n}{k}\mathcal{U}_n\left(\frac{k}{n}\left(\frac{n}{k}\overline{F}\left(xa_k\right)\right)\right)-\frac{n}{k}\overline{F}\left(xa_k\right)\right\},$$

and use weak approximation (5.7) to get

$$\sqrt{k}T_{n5}^{(1)}=\frac{\overline{G}\left(xa_k\right)}{\overline{G}\left(a_k\right)}\left(W_n\left(\frac{n}{k}\overline{F}\left(xa_k\right)\right)+o_{\mathbf{P}}\left(1\right)\left(\frac{n}{k}\overline{F}\left(xa_k\right)\right)^{\eta}\right).$$

By applying Proposition 5.1 twice (for $\overline{G}$ and $\overline{F}$), we get

$$\sqrt{k}T_{n5}^{(1)}=x^{-1/\gamma_2}W_n\left(\frac{n}{k}\overline{F}\left(xa_k\right)\right)+o_{\mathbf{P}}\left(x^{-1/\gamma_2-\eta/\gamma+\epsilon}\right).$$

For convenience, we set $h_n\left(s\right):=\frac{n}{k}\overline{F}\left(\mathbb{U}_G\left(\frac{1}{s\overline{G}(a_k)}\right)\right)$ to write

$$\sqrt{k}T_{n5}^{(2)}=-\int_0^{\overline{G}(xa_k)/\overline{G}(a_k)}\sqrt{k}\left(\frac{n}{k}\mathcal{U}_n\left(\frac{k}{n}h_n\left(s\right)\right)-h_n\left(s\right)\right)ds,$$

which by using weak approximation (5.7) equals

$$-\int_0^{\overline{G}(xa_k)/\overline{G}(a_k)}W_n\left(h_n\left(s\right)\right)ds+o_{\mathbf{P}}\left(1\right)\int_0^{\overline{G}(xa_k)/\overline{G}(a_k)}\left(h_n\left(s\right)\right)^{\eta}ds.$$

Observe that $\overline{F}\left(\mathbb{U}_G\left(1/\overline{G}\left(a_k\right)\right)\right)=k/n$, it follows that

$$h_n\left(s\right)=\overline{F}\left(\mathbb{U}_G\left(\frac{1}{s\overline{G}\left(a_k\right)}\right)\right)/\overline{F}\left(\mathbb{U}_G\left(\frac{1}{\overline{G}\left(a_k\right)}\right)\right).$$

Note that $\overline{F} \circ \mathbb{U}_G(1/\cdot) \in \mathcal{RV}_{(\gamma_2/\gamma)}$ near zero, then by the routine application of Proposition 5.1, we end up with

$$\int_0^{\overline{G}(xa_k)/\overline{G}(a_k)} (h_n(s))^\eta ds = O\left(x^{-\eta(1/\gamma_2+1/\gamma)+\epsilon}\right).$$

Hence, we showed that

$$\sqrt{k}T_{n5} = x^{-1/\gamma_2} W_n\left(\frac{n}{k}\overline{F}(xa_k)\right) - \int_0^{\overline{G}(xa_k)/\overline{G}(a_k)} W_n(h_n(s))\, ds + o_{\mathbf{P}}\left(x^{-\eta(1/\gamma_2+1/\gamma)+\epsilon}\right).$$

Observe now that, by using the mean value theorem, we get

$$\int_{x^{-1/\gamma_2}}^{\overline{G}(xa_k)/\overline{G}(a_k)} W_n(h_n(s))\, ds = \left(\frac{\overline{G}(xa_k)}{\overline{G}(a_k)} - x^{-1/\gamma_2}\right) W_n(h_n(g_n(x))),$$

where $g_n(x)$ is between $\overline{G}(xa_k)/\overline{G}(a_k)$ and x^{-1/γ_2}. It is easy to check that

$$h_n(g_n(x)) < (1+\epsilon x^\epsilon)\, x^{-1/\gamma}, \text{ for any } x \geq 1,$$

it follows that

$$\sup_{x\geq 1}\left|(h_n(g_n(x)))^{1/2} W_n(h_n(g_n(x)))\right| \leq \sup_{0\leq u\leq 1+\epsilon} |W_n(u)|,$$

which is stochastically bounded, therefore

$$\int_{x^{-1/\gamma_2}}^{\overline{G}(xa_k)/\overline{G}(a_k)} W_n(h_n(s))\, ds = O_{\mathbf{P}}(1)\,(h_n(g_n(x)))^{-1/2}\left|\frac{\overline{G}(xa_k)}{\overline{G}(a_k)} - x^{-1/\gamma_2}\right|,$$

uniformly on $x \geq 1$. By using the routine manipulations of Proposition 5.1, we show that

$$\overline{G}(xa_k)/\overline{G}(a_k) - x^{-1/\gamma_2} = o\left(x^{-1/\gamma_2+\epsilon}\right) \text{ and } (h_n(g_n(x)))^{-1/2} = O\left(x^{-1/(2\gamma)+\epsilon}\right),$$

thereby

$$\int_{x^{-1/\gamma_2}}^{\overline{G}(xa_k)/\overline{G}(a_k)} W_n(h_n(s))\, ds = o_{\mathbf{P}}\left(x^{-1/\gamma_2-1/(2\gamma)+\epsilon}\right) = o_{\mathbf{P}}(\varrho(x)),$$

because $0 < \eta < 1/2$. Next we show that

$$W_n\left(\frac{n}{k}\overline{F}(xa_k)\right) = W_n\left(x^{-1/\gamma}\right) + o_{\mathbf{P}}(\varrho(x)),$$

uniformly on $w \geq 1$. Let us fix $d > 0$ and set $\varrho_n(x) := \left|\frac{n}{k}\overline{F}(xa_k) - x^{-1/\gamma}\right|$ to write

$$\begin{aligned} &\mathbf{P}\left(\sup_{w\geq 1} x^{1/(2\gamma)-\epsilon}\left|W_n\left(\frac{n}{k}\overline{F}(xa_k)\right) - W_n\left(x^{-1/\gamma}\right)\right| > d\right) \\ =\ &\mathbf{P}\left(\sup_{w\geq 1} x^{1/(2\gamma)-\epsilon}|W_n(\varrho_n(x))| > d\right) = \mathbf{P}\left(|W_n(1)|\sup_{w\geq 1} x^{1/(2\gamma)-\epsilon}(\varrho_n(x))^{1/2} > d\right), \end{aligned}$$

which, by Markov's inequality, is less than or equal to $d^{-2}\sup_{w\geq 1} x^{1/(2\gamma)-\epsilon}(\varrho_n(x))^{1/2}$. Since $\varrho_n(x) = o\left(x^{-1/\gamma+\epsilon}\right)$, uniformly on $w \geq 1$, then the latter probability equals $o(1)$ as sought. Hence, we showed that

$$\int_0^{x^{-1/\gamma_2}} W_n(h_n(s))\, ds = \int_0^{x^{-1/\gamma_2}} W_n\left(s^{\gamma_2/\gamma}\right) ds + o_{\mathbf{P}}(\varrho(x)),$$

thus

$$\sqrt{k}T_{n5} = x^{-1/\gamma_2} W_n\left(x^{-1/\gamma}\right) - \int_0^{x^{-1/\gamma_2}} W_n\left(s^{\gamma_2/\gamma}\right) ds + o_{\mathbf{P}}(\varrho(x)).$$

By using a change of variables, the latter equation becomes

$$\sqrt{k}T_{n5} = x^{1/\gamma-1/\beta} W_n\left(x^{-1/\gamma}\right) + (1-\gamma/\beta)\int_0^{x^{-1/\gamma}} t^{\gamma/\beta-2} W_n(t)\, dt + o_{\mathbf{P}}(\varrho(x)). \tag{5.13}$$

It follows that, from (5.11), (5.12) and (5.13), that (5.6) is indeed true.

5.1.7 *Asymptotic behavior of* $\widetilde{\mathcal{B}}_n^{(2)}(x)$

It is easy to verify that

$$\widetilde{\mathcal{B}}_n^{(2)}(x) = \left(\frac{\overline{H}(xa_k)}{\overline{H}(a_k)} - x^{-1/\beta}\right)\left(\int_{a_k}^{\infty} \frac{\overline{G}(w)}{\overline{G}(a_k)} d\frac{F(w)}{\overline{F}(a_k)}\right).$$

By using inequality (5.10) (applied to $\overline{H}$), that for possibly different functions $\widetilde{A}_H$, with $\widetilde{A}_H(t) \sim A_H(t)$, as $t \to \infty$, and for each $0 < \epsilon < 1$, there exists $t_0 = t_0(\epsilon)$, such that for all $t \geq t_0$ and $x \geq 1$, we have

$$\left|\frac{\overline{H}(tx)/\overline{H}(t) - x^{-1/\beta}}{\widetilde{A}_H(t)} - x^{-1/\beta}\frac{x^{\rho_H/\beta}-1}{\rho_H\beta}\right| \leq \epsilon x^{-1/\beta+\epsilon}.$$

Thus by letting $t = a_k$, we write

$$\frac{\overline{H}(xa_k)}{\overline{H}(a_k)} - x^{-1/\beta} = x^{-1/\beta}\left(\frac{x^{\rho_H/\beta}-1}{\rho_H\beta} + o(x^{\epsilon})\right)\widetilde{A}_H(a_k).$$

uniformly on $x \geq 1$. Since $\widetilde{A}_H(a_k) \sim \mathbb{A}_H(n/k)$ then

$$\frac{\overline{H}(xa_k)}{\overline{H}(a_k)} - x^{-1/\beta} = x^{-1/\beta}\left(\frac{x^{\rho_H/\beta}-1}{\rho_H\beta} + o(x^{\epsilon})\right)\sqrt{k}\mathbb{A}_H(n/k).$$

On the other hand, we have

$$\int_{a_k}^{\infty} \frac{\overline{G}(w)}{\overline{G}(a_k)} d\frac{F(w)}{\overline{F}(a_k)} \to \frac{\beta}{\gamma},$$

it follows that $\widetilde{\mathcal{B}}_n^{(2)}(x) = x^{-1/\beta}\left(\frac{x^{\rho_H/\beta}-1}{\rho_H\gamma} + o(x^{\epsilon})\right)\mathbb{A}_H(n/k)$, uniformly on $x \geq 1$.

5.1.8 *Summarize*

Up to now we showed that

$$\sqrt{k}\left(\Delta_n^{(2)}(x) - (\beta/\gamma)x^{-1/\beta}\right) = \Theta_n(x) + \sqrt{k}\widetilde{\mathcal{B}}_n^{(2)}(x) + o_{\mathbf{P}}(\varrho(x)),$$

where

$$\Theta_n(x) := x^{1/\gamma-1/\beta}W_n\left(x^{-1/\gamma}\right) - \frac{\beta}{\gamma}x^{-1/\beta}W_n(1) + \left(1-\frac{\gamma}{\beta}\right)\int_0^{x^{-1/\gamma}} t^{\gamma/\beta-2}W_n(t)\,dt,$$

uniformly on $x \geq 1$. Recall that $\varrho(x) = x^{-\eta/\beta}$ and note that $\Theta_n(x) = O_{\mathbf{P}}(\varrho(x))$ and $\widetilde{\mathcal{B}}_n^{(2)}(x) = o_{\mathbf{P}}(\varrho(x))$, because $\mathbb{A}_H(n/k) = o(1)$, it follows that

$$\Delta_n^{(2)}(x) - (\beta/\gamma)x^{-1/\beta} = o_{\mathbf{P}}(\varrho(x)).$$

Let $0 < \nu < \eta < 1/2$ be sufficiently small, then

$$x^{\nu/\beta}\left(\Delta_n^{(2)}(x) - (\beta/\gamma)x^{-1/\beta}\right) = o_{\mathbf{P}}\left(x^{(\nu-\eta)/\beta+\epsilon}\right)$$

uniformly on $x \geq 1$. It follows that

$$\sup_{x\geq 1} x^{\nu/\beta}\left|\Delta_n^{(2)}(x) - (\beta/\gamma)x^{-1/\beta}\right| \overset{\mathbf{P}}{\to} 0$$

and $\Delta_n^{(2)}(1) \overset{\mathbf{P}}{\to} \beta/\gamma$, thus by (5.1) we get $\sup_{x\geq 1} x^{\nu/\beta}\left|D_n^{(2)}(x)\right| \overset{\mathbf{P}}{\to} 0$, as well, which gives (2.3) (for $i = 2$). Observe now that

$$\sqrt{k}\left(\Delta_n^{(2)}(1) - (\beta/\gamma)\right) = \Theta_n(1) + o_{\mathbf{P}}(1),$$

it follows form (5.1), that

$$\sqrt{k}\Delta_n^{(2)}(1)\,D_n^{(2)}(x) = \Theta_n(x) - x^{-1/\beta}\Theta_n(1) + \sqrt{k}\widetilde{\mathcal{B}}_n^{(2)}(x) + o_{\mathbf{P}}(\varrho(x)).$$

It is ready to check that $\Theta_n(x) - x^{-1/\beta}\Theta_n(1) \equiv \mathcal{L}_n^{(2)}(x)$, thus the weak approximation (2.4) (for $i = 2$) comes. This completes the proof of the theorem.

5.2 Proof of Theorem 2.2

Recall that

$$\widehat{\gamma}-\gamma=\int_1^\infty x^{-1}D_n^{(1)}(x)\,dx \text{ and } \widehat{\beta}-\beta=\int_1^\infty x^{-1}D_n^{(2)}(x)\,dx.$$

By applying respectively the two first results in Theorem 2.1, we easily show that $\widehat{\gamma}\overset{\mathbf{P}}{\to}\gamma$ and $\widehat{\beta}\overset{\mathbf{P}}{\to}\beta$, that we omits further details. To establish the asymptotic normality, let us first write

$$\widehat{\gamma}_1-\gamma_1=\frac{2\beta^2}{(\widehat{\gamma}-2\beta)(\gamma-2\beta)}(\widehat{\gamma}-\gamma)-\frac{\widehat{\gamma}^2}{\left(\widehat{\gamma}-2\widehat{\beta}\right)(\widehat{\gamma}-2\beta)}\left(\widehat{\beta}-\beta\right). \tag{5.14}$$

By making use of, respectively, two Gaussian approximations in Theorem 2.1 yields

$$\sqrt{k}(\widehat{\gamma}-\gamma)=\int_1^\infty x^{-1}\mathcal{L}_n^{(1)}(x)\,dx+\int_1^\infty x^{-1}\sqrt{k}\mathcal{B}_n^{(1)}(x)\,dx+o_{\mathbf{P}}(1)$$

and

$$\sqrt{k}\left(\widehat{\beta}-\beta\right)=\int_1^\infty x^{-1}\mathcal{L}_n^{(2)}(x)\,dx+\int_1^\infty x^{-1}\sqrt{k}\mathcal{B}_n^{(2)}(x)\,dx+o_{\mathbf{P}}(1).$$

By using an integration by parts with a change of variables, we end up with

$$\sqrt{k}(\widehat{\gamma}-\gamma)=\gamma\int_0^1 s^{-1}W_n(s)\,ds-\gamma W_n(1)+\frac{\sqrt{k}\mathbb{A}_F(n/k)}{1-\rho_F}+o_{\mathbf{P}}(1),$$

and

$$\begin{aligned}\sqrt{k}\left(\widehat{\beta}-\beta\right) &= (2\gamma-\beta)\frac{\gamma}{\beta}\int_0^1 s^{\gamma/\beta-2}W_n(s)\,ds-\gamma W_n(1)\\ &\quad+\left(\frac{\gamma}{\beta}-1\right)\frac{\gamma^2}{\beta}\int_0^1 s^{\gamma/\beta-2}W_n(s)(\log s)\,ds+\frac{\sqrt{k}\mathbb{A}_H(n/k)}{1-\rho_H}+o_{\mathbf{P}}(1).\end{aligned}$$

The previous two representations mean that $\sqrt{k}(\widehat{\gamma}-\gamma)$ and $\sqrt{k}\left(\widehat{\beta}-\beta\right)$ are asymptotically Gaussian rv's, which imply that

$$\sqrt{k}(\widehat{\gamma}-\gamma)=O_{\mathbf{P}}(1)=\sqrt{k}\left(\widehat{\beta}-\beta\right).$$

Then in view of (5.14) together with the consistency of $\widehat{\gamma}$ and $\widehat{\beta}$, we get

$$\sqrt{k}(\widehat{\gamma}_1-\gamma_1)=\frac{2\beta^2}{(\gamma-2\beta)^2}\sqrt{k}(\widehat{\gamma}-\gamma)-\frac{\gamma^2}{(\gamma-2\beta)^2}\sqrt{k}\left(\widehat{\beta}-\beta\right)+o_{\mathbf{P}}(1).$$

By assumptions, we have $\sqrt{k}\mathbb{A}_F(n/k)\to\lambda_F$ and $\sqrt{k}\mathbb{A}_H(n/k)\to\lambda_H$, it follows that

$$\sqrt{k}(\widehat{\gamma}_1-\gamma_1)=Z_{n1}+Z_{n2}+\mu+o_{\mathbf{P}}(1),$$

where

$$\frac{(\gamma-2\beta)^2}{2\beta^2}Z_{n1}:=\gamma\int_0^1 s^{-1}W_n(s)\,ds-\gamma W_n(1)$$

and

$$\begin{aligned}-\frac{(\gamma-2\beta)^2}{\gamma^2}Z_{n2} &:= (2\gamma-\beta)\frac{\gamma}{\beta}\int_0^1 s^{\gamma/\beta-2}W_n(s)\,ds-\gamma W_n(1)\\ &\quad+\left(\frac{\gamma}{\beta}-1\right)\frac{\gamma^2}{\beta}\int_0^1 s^{\gamma/\beta-2}W_n(s)(\log s)\,ds,\end{aligned}$$

with μ is as in (2.5). Note that both Z_{n1} and Z_{n2} are centred Gaussian rv's, then it remains to compute the second order moment of $Z_{n1}+Z_{n2}$. To this end, let us define the following quantities

$$\Delta_1(\rho):=\int_0^1 s^{\rho-2}W_n(s)\,ds,\ \Delta_2(\rho):=\int_0^1 s^{\rho-2}W_n(s)(\log s)\,ds,\ \Delta_3:=W_n(1).$$

Thereby, we write

$$\frac{(\gamma-2\beta)^2}{2\beta^2}Z_{n1}:=\gamma\Delta_1(1)-\gamma\Delta_3$$

and

$$-\frac{(\gamma-2\beta)^2}{\gamma^2}Z_{n2}:=(2\gamma-\beta)\frac{\gamma}{\beta}\Delta_1(\gamma/\beta)-\gamma\Delta_3+\left(\frac{\gamma}{\beta}-1\right)\frac{\gamma^2}{\beta}\Delta_2(\gamma/\beta).$$

By using elementary computations, we end up with the following expectations:

$$\mathbf{E}\left[\Delta_1^2(\rho)\right]=\frac{2}{\rho(2\rho-1)},\ \mathbf{E}\left[\Delta_2^2(\rho)\right]=\frac{2(4\rho-1)}{\rho^2(2\rho-1)^3},\ \mathbf{E}\left[\Delta_3^2\right]=1,$$
$$\mathbf{E}\left[\Delta_1(\rho)\Delta_2(\rho)\right]=\frac{1-4\rho}{\rho^2(2\rho-1)^2},\ \mathbf{E}\left[\Delta_1(\rho)\Delta_3\right]=\frac{1}{\rho},\ \mathbf{E}\left[\Delta_2(\rho)\Delta_3\right]=-\frac{1}{\rho^2}.$$

This gives

$$\mathbf{E}\left[Z_{n1}\right]^2=\frac{4\beta^4\gamma^2}{(\gamma-2\beta)^4},\ \mathbf{E}\left[Z_{n2}\right]^2=\frac{\beta\gamma^6\left(\beta^2-2\beta\gamma+2\gamma^2\right)}{(2\gamma-\beta)^3(\gamma-2\beta)^4}$$

and

$$\mathbf{E}\left[Z_{n1}Z_{n2}\right]=-\frac{2\beta^4\gamma^2}{(\gamma-2\beta)^4},$$

therefore

$$\begin{aligned}\mathbf{E}\left[\widehat{\gamma}_1-\gamma_1\right]^2&=\mathbf{E}\left[Z_{n1}\right]^2+\mathbf{E}\left[Z_{n2}\right]^2+2\mathbf{E}\left[Z_{n1}Z_{n2}\right]+o(1)\\&=\frac{\gamma^6\beta\left(\beta^2-2\beta\gamma+2\gamma^2\right)}{(2\gamma-\beta)^3(\gamma-2\beta)^4}+o(1),\end{aligned}$$

which completes the proof of the lemma.

6 APPENDIX B

Proposition 6.1. *Assume that $\overline{F}\in\mathcal{RV}_{(-1/\gamma_1)}$ and $\overline{G}\in\mathcal{RV}_{(-1/\gamma_2)}$. Then, for every $r,s\geq 0$, we have*

$$\frac{\mathbf{E}\left[\left(\overline{G}(X)\right)^r(\log(X/t))^s\mid X>t\right]}{\mathbf{E}\left[\left(\overline{G}(X)\right)^r\mid X>t\right]}\to\left(\frac{\gamma_1\gamma}{(1+r)\gamma_1-r\gamma}\right)^s\Gamma(s+1),\ \textit{as } t\to\infty.$$

Proof. Observe that

$$\frac{\mathbf{E}\left[\left(\overline{G}(X)\right)^r(\log(X/t))^s\mid X>t\right]}{\mathbf{E}\left[\left(\overline{G}(X)\right)^r\mid X>t\right]}=\frac{\mathcal{I}_t(s)}{\mathcal{I}_t(0)},$$

where

$$\mathcal{I}_t(s):=\int_t^\infty\left(\frac{\overline{G}(x)}{\overline{G}(t)}\right)^r(\log(x/t))^s\frac{dF(x)}{\overline{F}(t)}.$$

Let us decompose $\mathcal{I}_t(s)$ into the sum of

$$\mathcal{I}_{t,1}:=-\int_1^\infty\left\{\left(\frac{\overline{G}(tx)}{\overline{G}(t)}\right)^r-x^{-r/\gamma_2}\right\}(\log x)^s d\frac{\overline{F}(tx)}{\overline{F}(t)},$$
$$\mathcal{I}_{t,2}:=-\int_1^\infty x^{-r/\gamma_2}(\log x)^s d\left\{\frac{\overline{F}(tx)}{\overline{F}(t)}-x^{-1/\gamma}\right\}$$

and $\mathcal{I}_{t,3}:=-\int_1^\infty x^{-r/\gamma_2}(\log x)^s dx^{-1/\gamma}$. Next we show that both $\mathcal{I}_{t,1}$ and $\mathcal{I}_{t,2}$ tend to zero as $t\to\infty$. Indeed, let us write

$$|\mathcal{I}_{t,1}|\leq\int_1^\infty\left|\left(\frac{\overline{G}(tx)}{\overline{G}(t)}\right)^r-x^{-r/\gamma_2}\right|(\log x)^s d\frac{F(tx)}{\overline{F}(t)}.$$

Since $\overline{G} \in \mathcal{RV}_{(-1/\gamma_2)}$ then $\overline{G} \in \mathcal{RV}_{(-r/\gamma_2)}$ therefore by applying Proposition 5.1 yields

$$\left(\frac{\overline{G}(tx)}{\overline{G}(t)}\right)^r - x^{-r/\gamma_2} = o\left(x^{-r/\gamma_2+\epsilon}\right) = o(1), \text{ as } t \to \infty,$$

for every small $\epsilon > 0$ and uniformly on $x \geq 1$. It follows that

$$\mathcal{I}_{t,1} = o(1) \int_1^\infty (\log x)^s \, d\frac{F(tx)}{\overline{F}(t)}.$$

By using an integration by parts, we show that

$$\int_1^\infty (\log x)^s \, d\frac{F(tx)}{\overline{F}(t)} = \int_1^\infty \frac{\overline{F}(tx)}{\overline{F}(t)} d(\log x)^s .$$

Once again, from Proposition 5.1, $\overline{F}(tx)/\overline{F}(t) = (1 + o(x^\epsilon)) x^{-1/\gamma}$, then the previous integral becomes

$$\int_1^\infty (1 + o(x^\epsilon)) x^{-1/\gamma} d(\log x)^s .$$

It is clear that

$$\begin{aligned}\int_1^\infty x^{-1/\gamma} d(\log x)^s &= s\int_1^\infty (\log x)^{s-1} x^{-1/\gamma-1} dx \\ &= \gamma^s s \int_0^\infty v^{s-1} e^{-v} dv = \gamma^s s \Gamma(s),\end{aligned}$$

which, from the gamma function properties, is finite for any $s \geq 0$. This implies that

$$\int_1^\infty (1 + o(x^\epsilon)) x^{-1/\gamma} d(\log x)^s < \infty,$$

for any $s \geq 0$ and small $\epsilon > 0$, and therefore $\mathcal{I}_{t,1} = o(1)$. For the term $\mathcal{I}_{t,2}$ we use once gain an integration by parts with similar arguments to get $\mathcal{I}_{t,2} = o(1)$ as well. By using elementary analysis with a change of variables, we show that

$$\mathcal{I}_{t,3} = \frac{\Gamma(s+1)}{\gamma(r/\gamma_2 + 1/\gamma)^{s+1}},$$

thereby

$$\frac{\mathcal{I}_t(s)}{\mathcal{I}_t(0)} = \frac{\Gamma(s+1)}{(r/\gamma_2 + 1/\gamma)^s} + o(1), \text{ as } t \to \infty.$$

Finally, by replacing $1/\gamma_2$ by $1/\gamma - 1/\gamma_1$, we complete the proof of Proposition 6.1. ∎

Proposition 6.2. *Let $\mathcal{R}_n(s) := n^{-1} \sum_{i=1}^n \mathbb{I}(\xi_i \leq s)$, be the uniform empirical df pertaining to a sequence of iid rv's ξ_i, $i = 1, ..., n$ uniformly distributed on $(0,1)$. Then, for $n \geq 1$, we have*

$$\sup_{\xi_{1:n} \leq t \leq 1} \frac{t}{\mathcal{R}_n(t)} = O_{\mathbf{P}}(1) = \sup_{\xi_{1:n} \leq t \leq 1} \frac{\mathcal{R}_n(t)}{t},$$

where $\xi_{1:n} := \min_{1 \leq i \leq n}(\xi_i)$.

Proof. The proofs of the first two assertions may be found in [17] (pages 415 and 416, inequality 2). ∎

Proposition 6.3. *Let $k = k_n$ be an integer sequence satisfying $k \to \infty$ and $k/n \to 0$, then*

$$(i)\ \sqrt{k}\left(1 - \left(\frac{nU_{k:n}}{k}\right)^\alpha\right) = W_n(1) + o_{\mathbf{P}}(1).$$

If the second-order condition (2.1) (for $\overline{F}$) holds, then

$$(ii)\ \sqrt{k}\left(\left(\frac{X_{n-k:n}}{a_k}\right)^\alpha - 1\right) = \alpha\gamma W_n(1) + o_{\mathbf{P}}(1),$$

and

$$(iii)\ \sqrt{k}\left(1-\frac{\overline{F}\left(xX_{n-k:n}\right)}{\overline{F}\left(xa_k\right)}\right)=W_n\left(1\right)+o_{\mathbf{P}}\left(1\right),$$

uniformly on $x\geq 1$, *for every real* α.

Proof. Let us start by to prove assertion (i) for $\alpha=1$. Observe that

$$\sqrt{k}\left(1-nU_{k:n}/k\right)=\sqrt{k}\left(\frac{n}{k}\mathcal{U}_n\left(nU_{k:n}/k\right)-nU_{k:n}/k\right),$$

and from weak approximation (5.7), there exists a sequence of standard Wiener processes $W_n\left(s\right)$, such that

$$\sqrt{k}\left(\frac{n}{k}\mathcal{U}_n\left(nU_{k:n}/k\right)-nU_{k:n}/k\right)=W_n\left(nU_{k:n}/k\right)+o_{\mathbf{P}}\left(1\right).$$

Next we show that $W_n\left(nU_{k:n}/k\right)=W_n\left(1\right)+o_{\mathbf{P}}\left(1\right)$. To this end, let us

$$\epsilon_n:=\left|nU_{k:n}/k-1\right|$$

which tends to zero in probability. It is clear that for any fixed $d>0$, we have

$$\begin{aligned}&\mathbf{P}\left(\left|W_n\left(nU_{k:n}/k\right)-W_n\left(1\right)\right|>d\right)\\=\;&\mathbf{P}\left(\left|W_n\left(\epsilon_n\right)\right|>d\right)\leq\mathbf{P}\left(\sup_{0\leq s\leq\epsilon_n}\left|W_n\left(s\right)\right|>d\right).\end{aligned}$$

For sufficiently small $\epsilon>0$, the latter probability is less than or equal to

$$\mathbf{P}\left(\left|\sup_{0\leq s\leq\epsilon}\left|W_n\left(s\right)\right|\right|>d\right)+\epsilon\leq\mathbf{P}\left(\left|W_n\left(1\right)\right|>\epsilon^{-1/2}d\right)+\epsilon,$$

which by using Markov's inequality is $\left(d^{-2}+1\right)\epsilon$. This means that $W_n\left(nU_{k:n}/k\right)=W_n\left(1\right)+o_{\mathbf{P}}\left(1\right)$. To show assertion (i) for every real α, it suffices to use the mean value theorem and the fact that $nU_{k:n}/k=1+o_{\mathbf{P}}\left(1\right)$. For assertions (ii) and(iii), let us write

$$\begin{aligned}&\sqrt{k}\left(1-\frac{\overline{F}\left(xX_{n-k:n}\right)}{\overline{F}\left(xa_k\right)}\right)\\=\;&\sqrt{k}\left(\frac{nU_{k:n}}{k}-\frac{\overline{F}\left(xX_{n-k:n}\right)}{\overline{F}\left(xa_k\right)}\right)+\sqrt{k}\left(1-\frac{nU_{k:n}}{k}\right)\end{aligned}$$

and

$$\begin{aligned}&\sqrt{k}\left(\left(\frac{X_{n-k:n}}{a_k}\right)^{\alpha}-1\right)\\=\;&\sqrt{k}\left(\left(\frac{X_{n-k:n}}{a_k}\right)^{\alpha}-\left(\frac{nU_{k:n}}{k}\right)^{-\alpha\gamma}\right)+\sqrt{k}\left(\left(\frac{nU_{k:n}}{k}\right)^{-\alpha\gamma}-1\right).\end{aligned}$$

By using similar arguments with the second order condition of $\overline{F}$, we show that both first terms of right-hand of the previous equations tend to zero in probability. To achieve the proof it suffices to apply assertion (i), as sought. ∎

Proposition 6.4. *Assume* $\overline{F}\in 2\mathcal{RV}_{(-1/\gamma)}\left(A_F,\rho_F\right)$ *and* $\overline{G}\in 2\mathcal{RV}_{(-1/\gamma_2)}\left(A_G,\rho_G\right)$. *Then, for all large* x, *there exist constants* $c_1,c_2>0$, *such that*

$$\overline{F}\left(x\right)=\left(1+o\left(1\right)\right)c_1x^{-1/\gamma}\text{ and }\overline{G}\left(x\right)=\left(1+o\left(1\right)\right)c_2x^{-1/\gamma_2}.$$

Proof. See the proof of Lemma 7.1 in [3]. ∎