International Journal of Applied Mathematics and Simulation

Studying a Hidden Bifurcation and Finding Hopf Bifurcation with Generated New Saturated Function Series

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ABSTRACT: In this article, a hidden bifurcation of the multispiral chaotic attractor generated by the new saturated function series has been considered. The general shape of the chaotic attractors is described in terms of the number of spirals (also reffered to as multiscroll attractor) governed by integer parameters p and q. Due to the integer nature of the parameter, it is not possible to observe bifurcations from M spirals when the parameter is increased by two. However, by using the method of hidden bifurcations, an additional real parameter ε was introduced to observe such bifurcations. Additionally, this added parameter allowed us to find the Hopf bifurcation of the multispiral attractor generated by the new saturated function series transitioning from a stable state to a chaotic state. Furthermore, the Routh-Hurwitz criterion was used to study the stability of the original equilibrium point of the system.

Keywords: Saturated function series, hidden bifurcation, Hopf bifurcation, multiscroll.

MSC: 39A21, 39A28, 39A33.

1 Introduction

In nonlinear dynamics, chaotic systems and their dynamical characteristics are fascinating subjects (physical and engineering systems, climate models and global weather patterns and biological systems...). Such dynamical systems produce significantly differing results for small differences in initial conditions (e.g., rounding errors in numerical computation), making long-term behavior prediction generally difficult. In the last forty years, the scientific, mathematical, and engineering communities have devoted a great deal of attention to the study of chaos, a highly fascinating and complicated nonlinear phenomenon [1], [2], [3], [4].

Dynamical behavior can be effectively explained by the bifurcation theory [5]. When a parameter is changed, the dynamics of bifurcations of arbitrary invariant sets of dynamical systems seem more appealing and complex [5]. Hopf bifurcation, sometimes called Poincare-Andronov-Hopf bifurcation, is the local birth or death of a periodic solution (self-excited oscillation) from an equilibrium as a parameter reaches a critical point [6].

Moreover, while most of these multiscroll generations have been known for a long time, bifurcation theory has only lately been applied to their study [7]. They have also been identified for hidden attractors [8] in the situation of infinitely many equilibria, as well as in the case where equilibrium points exist. The number of scrolls (or spirals) for every multi scroll that is currently known is a fixed integer which is

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Communicated Editor: Khaled Zennir
Manuscript received Feb 20, 2024; revised Apr 17, 2024; accepted Mai 01, 2024; published Dec 07, 2024.

depends on one or more discrete characteristics [9].
In 2015, Menacer et al., introduced a hidden bifurcation theory and producing multiscrolls in a family of systems (6) with a continuous bifurcation parameter, modified the paradigm of discrete parameters [10]. This technique was discovered from the hidden attractor theory, first presented by Leonov et al. [12], [13], [14], [15], [16], which constitutes the foundation for this hidden bifurcation theory. We applied this technique to the $1-D$ multiscroll chaotic attractors generated by saturated function series [17]. A saturated function series was proposed for generating multi-scroll chaotic attractors, including 1-D *n*-scroll, 2-D *n* × *m*-grid scroll, and 3-D *n* × *m* × *l*-grid scroll chaotic attractors [19], [20].
This paper provides two new findings: first, we examined a hidden bifurcation in 1-D multiscroll chaotic attractors created by a new saturated function series. In comparison with previous results [10], [19] we found the difference in behavior and form of spirals; second, we determined the Hopf bifurcation and stability of the origin equilibrium point E_0 concerning ε and we identified a critical point for both. After a lot of calculation, we noticed that Hopf bifurcation was determined in this case only with these values set to me for $\alpha = t_1 = 0.72, \beta = \gamma = 0.8, k = 10, h = 20$.

This paper is organized as follows: In Section 2, the model of multiscroll chaotic attractors generated by the new saturated function series proposed is studied. In Section 3, the localization technique introduced for hidden bifurcation in multiscroll chaotic attractors generated by new saturated function series. In Section 4, Hopf bifurcation and stability of the origin equilibrium point E_0 for ε. Finally, in secion 5, we have a concluding comments. Appendix A presents the technique of Leonov et al., for seeking a hidden attractor.

2 Design Multispiral Chaotic Attractors from Saturated Function Series.

One of the fundamental PWL circuits is the saturated circuit, which is widely known. Saturated circuit characteristics are effectivly the PWL models for operational amplifiers [7]. This study presents a multi-piecewise non-linear saturated series model [17], which has the following expression:

$$\begin{cases} \dot{x} = y \\ \dot{y} = z \\ \dot{z} = -\alpha x - \beta y - \gamma z + t_1 g(x;k;h;p;q), \end{cases} \tag{1}$$

where

$$g(x;k;h;p;q) = \begin{cases} y_{1,k} \;\; if \; x > qh+1 \\ y_{2,k,i} \;\; if \; |x-ih| \leq 1 \\ \qquad -p \leq i \leq q \\ y_{3,k,i} \;\; if \;\; l_{1,i} < x < l_{2,i} \\ \qquad -p < i < q-1 \\ y_{4,k} \;\; if \; x < -ph-1, \end{cases} \tag{2}$$

with
$l_{1,i} = ih+1$ and $l_{2,i} = (i+1) \times h - 1$,
$y_{1,k} = (2q+1)\,k$, $y_{2,k,i} = k\,(x-ih) + 2ik$,
$y_{3,k,i} = (2i+1)\,k$ and $y_{4,k} = -\,(2p+1)\,k$.
Parameters p, q, h and k are integers, and $\alpha, \beta, \gamma, t_1$ are real numbers. This article aims to examine the attractors' overall form and global geometric characteristics, which are expressed in terms of the number of spirals, a phenomenon referred to as a hidden bifurcation [11]. Hopf bifurcations [5]. This work identifies structurally chaotic attractors with fixed real parameter values of $\alpha = t_1 = 0.72, \beta = \gamma = 0.8, k = 10$ and $h = 20$ (see Fig. 1 and see Fig. 2). The following formula determines the number M of spirals based on two integer inputs, p and q:

$$M = p + q + 2. \tag{3}$$

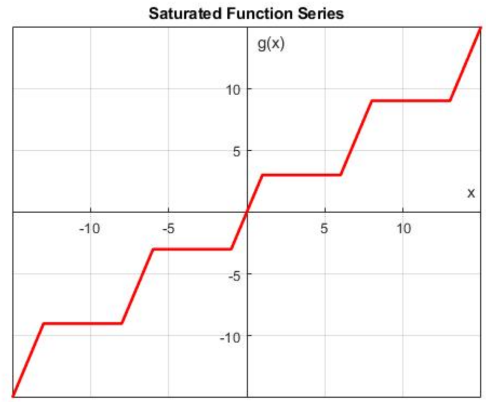


Fig. 1: **Proposed Saturated function series with $k = 3$, $h = 7$, $p = 2$, $q = 2$**

3 Hidden Bifurcations Revealing Technique

A distinctive technique for identifying hidden bifurcations was presented by Menacer et *al.* [5] to overcome this issue. This technique builds on the concept of Leonov and Kuznetsov [8] for examining hidden attractors (i.e., homotopy and numerical continuation, see Appendix A). This method is new when applied to multiscroll chaotic attractors from saturated function series. In this section, we briefly review the process, where the parameters values are fixed at $\alpha = t_1 = 0.72, \beta = \gamma = 0.8, k = 10, h = 20.$
Rewrite system (1)-(2) to the form:

$$\frac{dx}{dt} = Fx + \eta\Psi(\delta^T x), \qquad x \in \mathbb{R}.^3. \tag{4}$$

Where

$$F = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha & -\beta & -\gamma \end{pmatrix}, \quad \eta = \begin{pmatrix} 0 \\ 0 \\ t_1 \end{pmatrix}, \quad \delta = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \tag{5}$$

Presenting the coefficient k^* and small parameter ε, and describe system (4) as

$$\frac{dx}{dt} = F_0 x + \eta\varepsilon\varphi(\delta^T x), \tag{6}$$

where

$$F_0 = F + k^*\eta\delta^T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ k^*t_1 - \alpha & -\beta & -\gamma \end{pmatrix},$$

$$\rho_{1,2}^{F_0} = \pm i\omega_0, \quad \rho_3^{F_0} = -d.$$

By nonsingular linear transformation $X = SY$ system (6) is became to the form

$$\frac{dy}{dt} = Hy + B\varepsilon\varphi(c^T y), \tag{7}$$

where

$$H = \begin{pmatrix} 0 & -\omega_0 & 0 \\ \omega_0 & 0 & 0 \\ 0 & 0 & -d \end{pmatrix}, \quad B = \begin{pmatrix} b_1 \\ b_2 \\ 1 \end{pmatrix}, \quad c = \begin{pmatrix} 1 \\ 0 \\ -h \end{pmatrix}. \tag{8}$$

The transfer function $W_H(s)$ of system (7) can be presented as

$$W_H(s) = \frac{-b_1 s + b_2\omega_0}{s^2 + \omega_0^2} + \frac{h}{s + d}. \tag{9}$$

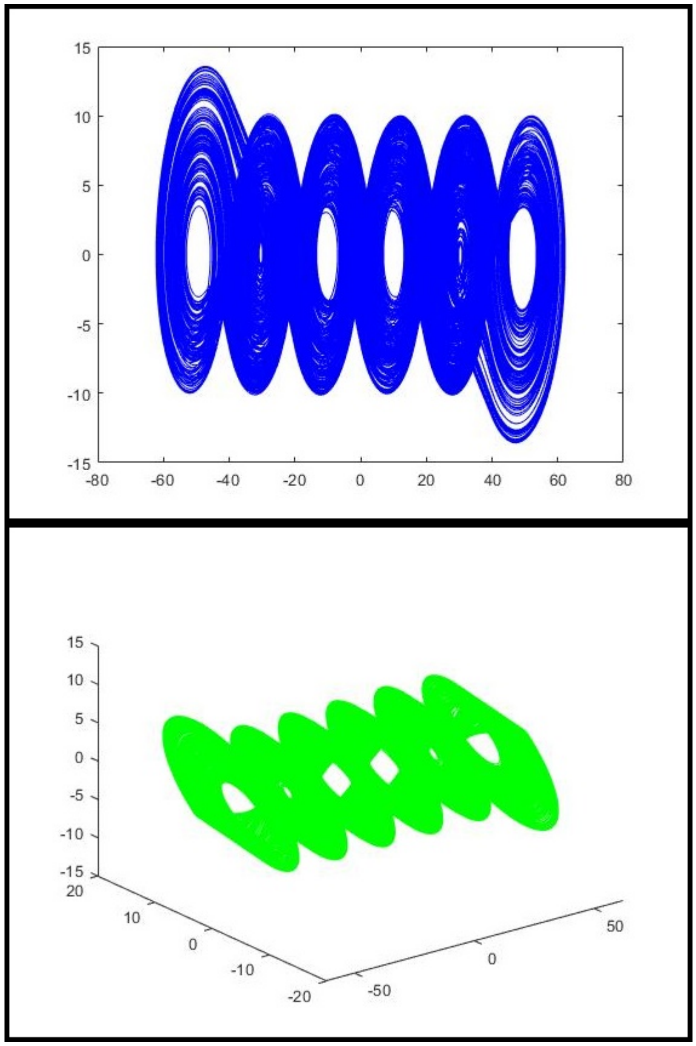


Fig. 2: (a) The 6-spiral attractor generated by equation (1) and (2) for $p = q = 2$ 3-projection into the plane (x, y), (b) The 6-spiral attractor generated by equation (1) and (2) for $p = q = 2$ $3-$ projection into the plane (x, y, z).

Also, utilizing the equality of transfer functions of systems (6) and system (7), we obtain:

$$W_{F_0}(s) = \delta^T (F_0 - sI)^{-1} \eta. \tag{10}$$

This implies the following relations:

$$\begin{aligned} k^* &= \tfrac{\alpha - \omega_0^2 d}{t_1}, \\ d &= c, \\ h &= \tfrac{-t_1}{\omega_0^2 + d^2} = b_1, \\ b_2 &= \tfrac{-\gamma_1}{\omega_0(\omega_0^2 + d^2)}. \end{aligned} \tag{11}$$

Since system (6) can be debilitated to the form (7) by the non-singular linear transformation defined in (A), the following relations can be acquired:

$$H = S^{-1} F_0 S,\ B = S^{-1}\eta,\ c^T = \delta^T S. \tag{12}$$

To solve these matrix equations, we obtain the following transformation matrix :

$$S = \begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{pmatrix},$$

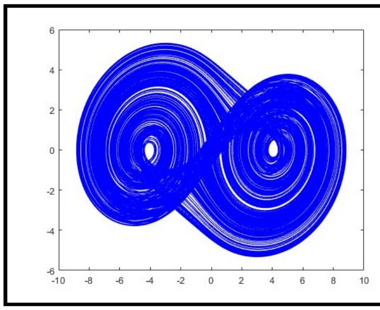


Fig. 3: 2 spiral for ε=0.35

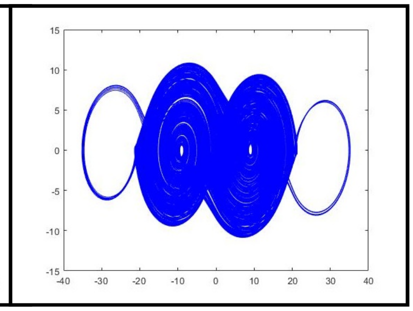


Fig. 4: 4 spirals for ε=0.94

with

$$\begin{array}{lll} S_{11} = 1, & S_{12} = 0\,, & S_{13} = -h \\ S_{21} = 0, & S_{22} = -\omega_0, & S_{23} = dh \\ S_{31} = -\omega_0^3, & S_{32} = 0, & S_{33} = d^2h. \end{array} \quad (13)$$

3.1 Numerical Results of the Hidden Bifurcation Technique

Using (A), with a sufficiently small ε we computed initial data for the first step of multistage localization procedure

$$X(0) = SY(0) = S \begin{pmatrix} \varsigma_0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \varsigma_0 S_{11} \\ \varsigma_0 S_{21} \\ \varsigma_0 S_{31} \end{pmatrix}. \quad (14)$$

For system (4), this gives the initial data

$$X^0(0) = (x^0(0) = \varsigma_0,\ y^0(0) = 0,\ z^0(0) = -\varsigma_0\omega_0^3.). \quad (15)$$

The localization process outlined previously is now applied to system 1 with numerous spiral attractors. In order to accomplish this, we calculate a harmonic linearization coefficient and the initial frequency ω_0 as described in the Appendix:

$$\omega_0 = 0.86, \quad k^* = 0.32. \quad (16)$$

Next, we compute the solutions to system (6) with the nonlinearity $\varepsilon\varphi(x) = \varepsilon(\psi(x) - k_1 x)$. To do this, we start at the beginning with step 0.35 and increase ε successively from the value $\varepsilon = 0.35$ to $\varepsilon = 1$. The starting data for the solutions for increasing values of ε, as shown in Table 1, is obtained via (15). So, from the Table 1, we obtain the solutions $X^1(0)$ with one scroll to $X^4(0)$ (See Fig. 3 to Fig 6). In each figure, there is a variant an even number of spirals in the attractor. The number of spirals increases by 2 at each step as shown on Table 2 from 2 to 6 spirals. The values of ε in this table are totally the values of bifurcation points.

Values of ε	$X^i(0)$	x_0	y_0	z_0
0.42	$X^1(0) = X^1(t_{max})$	-0.57	0	0.1252
0.94	$X^2(0) = X^2(t_{max})$	-13.8295	1.7315	5.3924
0.98	$X^3(0) = X^3(t_{max})$	27.1245	3.2587	-8.2567
1	$X^4(0) = X^4(t_{max})$	-1.1235	-2.1587	-7.2025

Table 1: Initial data according to the values of ε

Table 2: Values of the parameter ε at the bifurcation points for $p = q = 2$

Values of ε	0.35	0.94	0.98	1
Number of spirals	2 spiral	4 spirals	6 spirals	6 spirals

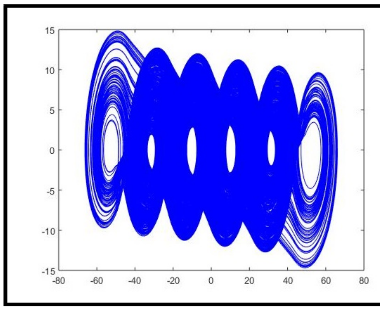


Fig. 5: 6 spirals for ε=0.98

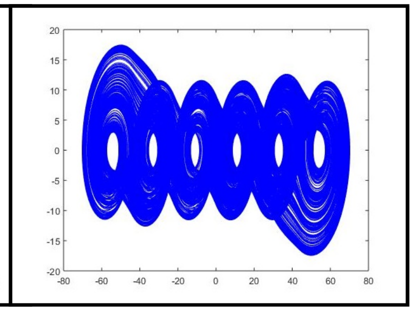


Fig. 6: 1 spirals for ε=1

4 Hopf Bifurcation and Stability of the Origin Equilibrium Point E_0 with Respect to ε

4.1 Stability of the Origin Equilibrium Point E_0 with Respect to ε

In the system (1)-(2) we have $2(p+q)+3$ equilibrium points. They have a positive eigenvalue and a pair of complex eigenvalues with negative real parts. This means that all equilibria of the system are saddle points [17]. Using the conditions established by Routh-Hurwitz [18], we examine the stability of the equilibrium point E_0 in relation to the epsilon of the system (6). In [11], Menacer et al. present the idea of hidden bifurcation in the Chua system by including a new parameter, epsilon, that regulates the spiral number. The number of scrolls reduces as ε increases from 0 to 1. The following polynomial yields the eigenvalues equation corresponding to this equilibrium point:

$$P(s) = s^3 + a_1 s^2 + a_2 s + a_3. \tag{17}$$

Using the result of the Routh-Hurwitz conditions, where the necessary and sufficient condition for the equilibrium point E to be locally asymptotically stable is $a_1 > 0$, $a_3 > 0$ and $a_1 \times a_2 - a_3 > 0$.

In our study, we study the stability and Hopf bifurcation with respect to the parameter ε and the parameters values are $\alpha = t_1 = 0.72, \beta = \gamma = 0.8, p = q = 2, k^* = 0.32$. An equilibrium point of system (1) independent of epsilon is the origin $E_0(0,0,0)$. The evaluation of the Jacobian matrix at the equilibrium point $E_0(0,0,0)$ is:

$$J_{E_0} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha k^* + \alpha\varepsilon(k^* + k) & -\beta & -\gamma \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.2304 + \varepsilon 6.7104 & -0.8 & -0.8 \end{pmatrix}.$$

Its characteristic polynomial is:

$$P(s) = s^3 + 0.8s^2 + 0.8s + (0.2304 + \varepsilon 6.7104).$$

The necessary and sufficient requirement for the equilibrium point is stated in the Routh-Hurwitz criteria. E_0 to be stable is $0.0343 < \varepsilon < 0.1297$.

Proof. we applied the Routh-Hurwitz criteria for origin equilibrium point (0,0,0) we found:

$$\text{First condition} : a_1 = 0.8 > 0 \tag{18}$$

$$\text{Second condition} : a_3 = 0.2304 + \varepsilon 6.7104 > 0 \Longrightarrow \varepsilon > 0.0.0343; \tag{19}$$

$$\text{Third condition} : a_1 \times a_2 - a_3 = 0.64 - 6.7104\varepsilon > 0 \Longrightarrow \varepsilon < 0.1297 \tag{20}$$

so E_0 to be stable when: $0.0343 < \varepsilon < 0.1297$. □

Remark

For the special case $\boldsymbol{\varepsilon = 0}$ **:** the system (1) becomes linear so the attractor as a limit cycle unstable see the figure below :

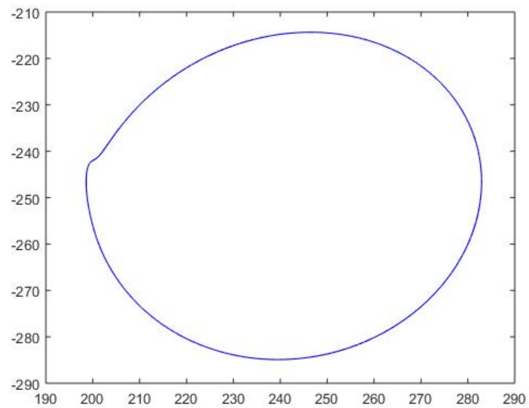


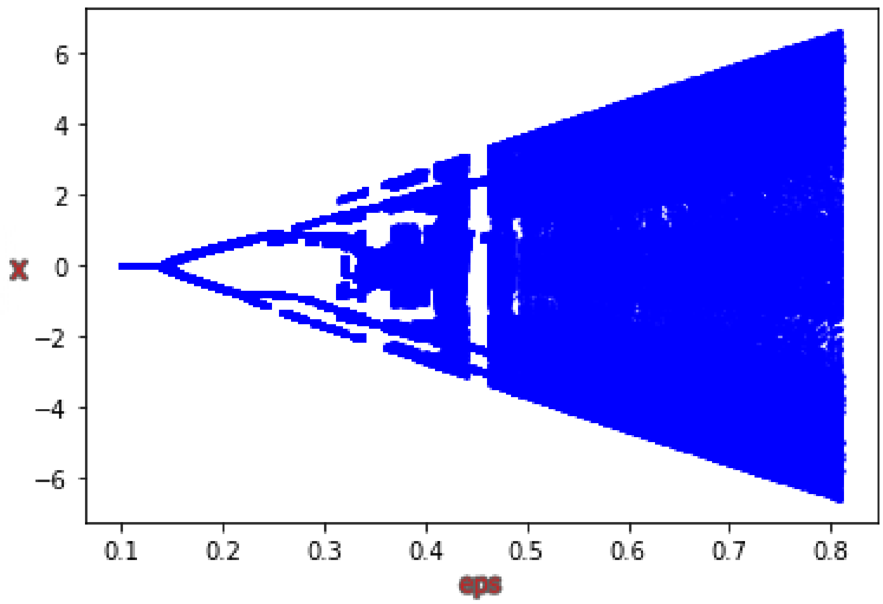
Fig. 7: **The system's attractor (1), in which $\varepsilon = 0$: limit of an unstable cycle.**

4.2 Analysis of Hopf Bifurcation at the Origin Equilibrium Point E_0 Regarding ε

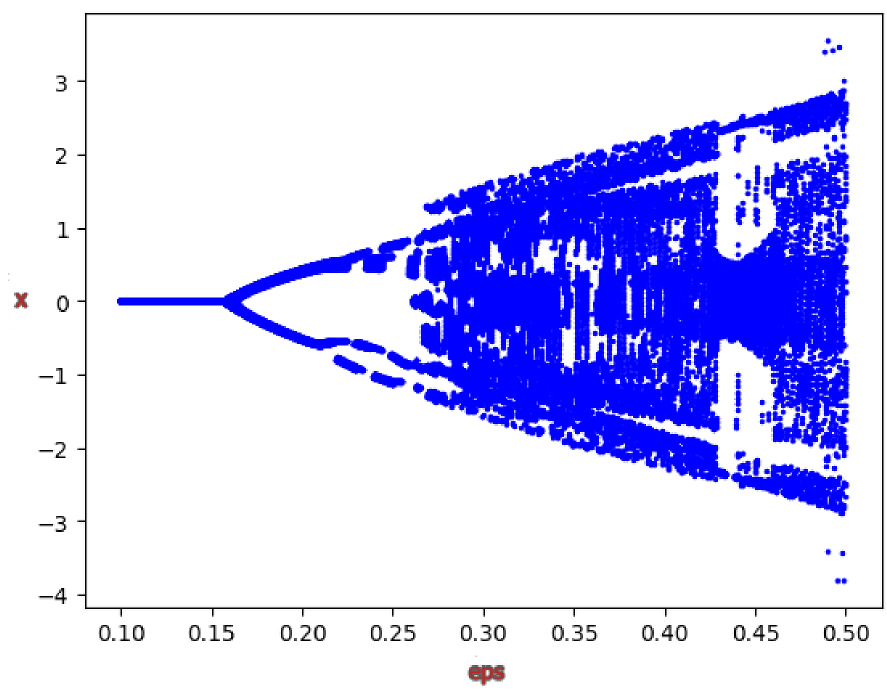
The system (1) attached by the formula (2) has $2(p+q)+3$ equilibrium points which we find by comparing the right sides of the system to zero and which are given in [17]. We studied Hopf bifurcation at the point $(0,0,0)$ with the values $\alpha = t_1 = 0.72, \beta = \gamma = 0.8, p = q = 2$. The conditions of system (1) with $p = q = 2$, to undergo a Hopf bifurcation at the equilibrium point $E(0,0,0)$ when $\varepsilon = \varepsilon^*$
-The Jacobian matrix has two complex-conjugate eigenvalues $s_{1,2} = \Theta(\varepsilon) \pm i\omega(\varepsilon)$ and one real $s_3(\varepsilon)$,
-$\Theta(\varepsilon^*) = 0$, and $s_3(\varepsilon^*) \neq 0$,
-$\omega(\varepsilon^*) \neq 0$,
-$\frac{d\Theta}{d\varepsilon}|_{\varepsilon=\varepsilon^*} \neq 0$.

Proposition 1. *The system (1) undergoes a Hopf bifurcation at $E(0,0,0)$, when the parameter ε crosses the critical values $\varepsilon^* = 0.16039$.*

Proof. For the first condition : $\Theta(0.16039) = 0$, and $s_3(0.16039) = -0.89444 \neq 0$,
For the second condition: $\omega(0.16039) = -0.80002 \neq 0$
For the last condition : $\frac{d\Theta}{d\varepsilon}|_{\varepsilon=0.16039} = 0.46988 \neq 0$. □



(a)



(b)

Fig. 8: **The results of Hopf bifurcation analysis (a): The bifurcation diagram for the critical point** $\varepsilon^* = 0.16039$ **, (b) : Clarify The bifurcation diagram for the critical point** $\varepsilon^* = 0.16039$**.**

5 CONCLUSION

This article examines hidden bifurcations of the multispiral Chaotic attractor produced by the newly discovered saturation function series. The number of spirals, or multiscroll attractor, determined by the integer parameters p, q, has been used to characterize the overall shape of the chaotic attractors. When this parameter is increased by two, bifurcations from M spirals cannot be observed because of its integer character. Nevertheless, an extra real parameter ε was added in order to observe such bifurcations using the hidden bifurcation approach. Additionally, the Hopf bifurcation of the multispiral attractor produced by the new saturated function series from a stable state to a chaotic state may be found thanks to this additional parameter. Furthermore, the stability of the system's initial equilibrium point was examined using the Routh-Hurwitz criteria. In our futur works, we will provide to find a hidden attractors and hidden bifurcations in new systems.

APPENDIX A
TECHNIQUE OF LEONOV ET AL FOR SEEKING A HIDDEN ATTRACTOR

The technique for seeking attractors of multidimensional nonlinear dynamical systems with scalar non-linearity was proposed by Leonov [12] and Leonov et al. [8], [13], [14], [15], [16]. Their technique is based on numerical continuation: a series of linked systems is built such that, for the first system (the starting system), the initial data for the numerical computation of a potential oscillating solution (the starting oscillation) can be obtained analytically. The proposed technique is extended in [16], [10] to the system of the form

$$\dot{U} = PX + qF(r^T X), \quad X \in \mathbb{R}^n, \tag{21}$$

where q, r are constant n-dimensional vectors, $F(\sigma)$ is a continuous piece- wise differential function reaching the condition $F(0) = 0$, and T implies transpose operation. P is a constant $n \times n$-matrix.

Here, we outline their technique for the simplified case when $n = 3$. Thus, we take into consideration the equation.

$$\dot{X} = PX + qF(r^T X), \quad U \in \mathbb{R}^3, \tag{22}$$

where $F(\sigma)$ is a continuous nonlinear function.

They then define a coefficient of harmonic linearization ϑ (suppose that such ϑ exists) in such a way that the matrix

$$P_0 = P + \vartheta q r^T, \tag{23}$$

of the linear system

$$\dot{X} = PX, \tag{24}$$

has a pair of purely imaginary eigenvalues $\pm i\omega_0, (\omega_0 > 0\ )$ and the other eigenvalue has negative real part. In practice, to determine ϑ and ω_0 they use the transfer function $W(\tau)$ of system(21)

$$W(\tau) = r(P - \tau I)^{-1} q, \tag{25}$$

where τ is a complex variable and I is a unit matrix. The number $\omega_0 > 0$ is determined from the equation $\mathrm{I}\varsigma\, W(i\omega_0) = 0$ and ϑ is calculated by the formula $\vartheta = -\operatorname{Re} W(i\omega_0)^{-1}$.

Therefore, system (21) can rewrite as

$$\dot{X} = P_0 X + qf(r^T X), \quad X \in \mathbb{R}^3, \tag{26}$$

where $f(\sigma) = F(\sigma) - \vartheta\sigma$.

Following that, they introduce a finite sequence of continuous functions $f^0(\varsigma)$, $f^1(\varsigma)$, ..., $f^m(\varsigma)$ in such a way that the graphs of neighboring functions $f^j(\varsigma)$ and $f^{j+1}(\varsigma)$, $(j = 0, ..., m-1)$ in a sense, slightly differ from each other, the function $f^0(\varsigma)$ is small and $f^m(\varsigma) = f(\varsigma)$. The function $f^0(\varsigma)$, allows the application the method of harmonic linearization (describing function method) to the system

$$\dot{X} = P_0 X + q f^0(r^T X), \quad X \in \mathbb{R}^3, \tag{27}$$

if the stable periodic solution $X^0(t)$ close to harmonic one is determined. Then for the localization of an attractor of the original system (26), one can follow numerically the transformation of this periodic solution (a starting oscillating is an attractor, not including equilibrium, denoted further by A_0) simply increasing j.

By non singular linear transformation S $(X = SZ)$ the system (27) can be reduced to the form

$$\begin{cases} \dot{z}_1(t) &= -\omega_0 z_2(t) + b_1 g^0(z_1(t) + c_3^T z_3(t)) \\ \dot{z}_2(t) &= \omega_0 z_1(t) + b_2 g^0(z_1(t) + c_3^T z_3(t)) \\ \dot{z}_3(t) &= a_3 z_3(t) + b_3 g^0(z_1(t) + c_3^T z_3(t)) \end{cases}, \tag{28}$$

where $z_1(t)$, $z_2(t)$, $z_3(t)$ are scalar values, a_3, b_1, b_2, b_3, c_3 are real numbers and $a_3 < 0$.

The describing function H is defined as

$$H(\varsigma) = \int_0^{\frac{2\pi}{\omega_0}} g(\cos(\omega_0 t)\varsigma)\cos(\omega_0 \varsigma) dt. \tag{29}$$

Theorem 2. *[8] If it can be found a positive ς_0 such that*

$$H(\varsigma_0) = 0, \quad b_1 \frac{dH(\varsigma)}{d\varsigma} \mid_{\varsigma=\varsigma_0} < 0,$$

has a stable periodic solution with initial data $X^0(0) = S(z_1(0), z_2(0), z_3(0))^T$ at the initial step of algorithm one has $z_1(0) = \varsigma_0 + O(\varepsilon)$, $z_2(0) = 0$, $z_3(0) = O_{n-2}(\varepsilon)$, where $O_{n-2}(\varepsilon)$ is an $(n-2)$-dimensional vector such that all it's components are $O(\varepsilon)$.

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