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Relationship between Sublinear Operators and their Subdifferentials for Certain Classes of Lipschitz Summability

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ABSTRACT: Let $\mathcal{SB}(X, Y)$ be the set of all bounded sublinear operators from a Banach space X into a complete Banach lattice Y; which is a pointed convex cone not salient in $\mathrm{Lip}_0(X, Y)$. In this paper, we are interested in studying the relationship between T and its subdifferential ∇T (the set of all bounded linear operators $u : X \longrightarrow Y$ such that $u(x) \leq T(x)$ for all x in X); concerning certain notions of Lipschitz summability. We also answer negatively a question posed previously concerning this type of relation in the linear case. For this, we introduce and study a new concept of summability in the category of Lipschitz operators, which we call "*super Lipschitz p-summing operators*". We prove some characterizations in terms of a domination theorem and some properties of this notion.

Keywords: Banach lattice, Lipschitz p-dominated operator, Lipschitz p-summing operator, p-summing operator, sublinear operator

MSC: Primary 46B25, 46T99; Secondary 47H99, 47L20

1 Introduction

The notion of p-summing linear operators goes back to Grothendieck in the 1950s, but just in 1967 and 1968, the classical works of Pietsch [18] and Lindenstrauss-Pełczyński [14] clarified Grothendieck's precious ideas and contributed clearly to the vigorous development of this notion. Recently, Lipschitz versions of different types of summing linear operators were investigated by several authors such as [4], [6], [7], [10], [19], [17] and [20] among others. The first paper is due to Farmer and Johnson [11]. They introduced the notion of Lipschitz p-summing operators and showed that is really a good generalization of the concept of linear p-summing operators [11, Theorem 2]. This notion marked the beginning of the theory of nonlinear summability. Motivated by the importance of this theory, several authors have developed and studied different concepts of summability. Chen and Zheng introduced in [7] (strongly) Lipschitz p-integral and p-nuclear operators. In [4] Chávez-Domínguez introduced the notion of Lipschitz (r, p, q)-summing operators and Lipschitz (q, p)-mixing in [5]. Independently, Yahi, Achour and Rueda [20] and Saadi [19] introduced and studied the class of Lipschitz strongly p-summing operators. The first authors introduced the notion of summing Lipschitz conjugates and (p, σ)- summability with an appropriate factorization. They characterized also those Lipschitz operators whose Lipschitz conjugates are absolutely p-summing.

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In the present work, we study the set of bounded sublinear operators which is a positive cone in $\mathrm{Lip}_0(X,Y)$. We shall consider the following situation: let X be a Banach space, Y be a complete Banach lattice and T be a bounded sublinear operator from X into Y (i.e., positively homogeneous and subadditive). We denote by ∇T the set of u in $\mathcal{B}(X,Y)$ (the Banach space of all bounded linear operators from X into Y) such that $u \leq T$. The aim of this work is to study the relation between T and its subdifferential ∇T concerning the notion of Lipschitz p-summing and other classes of summability. In this article, we show this relationship for some categories of summability.

The paper is organized as follows.

After the introduction, in section 1, we recall some basic definitions and properties concerning Banach lattices and sublinear operators.

In section 2, we study the Lipschitz p-summing sublinear operators and we introduce the class of Lipschitz super p-summing sublinear operators. We characterize this type of operators by giving a domination theorem. Also, we give some properties concerning this class. We study the relationship between T and u in ∇T concerning this type of summability and we answer negatively a question posed in [2] relative to the linear case, which is: T is p-summing if, and only if, u is p-summing for all u in ∇T? The first implication was shown in [2] and the reciprocal remained open. Here we answer this negatively.

Finally, we look in section 3 at the other types of summability. We are interested to the notion of Lipschitz p-dominated operators introduced in [7]. We show that if T is Lipschitz p-dominated sublinear operator, then $\nabla T \subset \Pi_p(X,Y)$. We prove in Proposition 4.3 below, if T is in $\mathcal{D}^L_{st,p}(X,Y)$ (the space of Lipschitz strongly p-summing operators) for $1 < p \leq \infty$, we have u positive strongly p-summing for all u in ∇T and hence u^* is positive p^*-summing with $\pi^+_{p^*}(u^*) \leq 2d^L_{st,p}(T)$. We also demonstrate that u is positive p-summing for all u in ∇T $(1 < p \leq 2)$ whenever T is in $\mathcal{D}^L_{st,p}(X, L_2(\Omega,\mu))$ with $\pi^+_p(u) \leq C d^L_{st,p}(T)$.

2 Definitions, notations and properties

Unless otherwise stated X, Y, Z will always real Banach spaces. As customary, $\mathcal{B}_X$ denotes the closed unit ball of X and X^* its topological linear dual. Now, we are going to introduce some terminology concerning the Banach lattices. For more details, the interested reader can consult the references [15], [16].
We recall the abstract definition of Banach lattice. Let X be a Banach space. If X is a vector lattice and $\|x\| \leq \|y\|$ whenever $|x| \leq |y|$ ($|x| = \sup\{x, -x\}$) we say that X is a Banach lattice. If the lattice is complete, we say that X is a complete Banach lattice. Note that this implies obviously that for any $x \in X$ the elements x and $|x|$ have the same norm. We denote by $X_+ = \{x \in X : x \geq 0\}$. An element x of X is positive if $x \in X_+$.
The dual X^* of a Banach lattice X is a complete Banach lattice endowed with the natural order

$$x_1^* \leq x_2^* \Longleftrightarrow \langle x_1^*, x\rangle \leq \langle x_2^*, x\rangle, \quad \forall x \in X_+ \tag{2.1}$$

where $\langle .,.\rangle$ denotes the bracket of duality.
By a sublattice of a Banach lattice X we mean a linear subspace E of X so that $\sup\{x,y\}$ belongs to E whenever $x, y \in E$. The canonical embedding $i : X \longrightarrow X^{**}$ such that $\langle i(x), x^*\rangle = \langle x^*, x\rangle$ of X into its second dual X^{**} is an order isometry from X onto a sublattice of X^{**}, see [15, Proposition 1.a.2]. If we consider X as a sublattice of X^{**} we have for $x_1, x_2 \in X$

$$x_1 \leq x_2 \Longleftrightarrow \langle x_1, x^*\rangle \leq \langle x_2, x^*\rangle, \quad \forall x^* \in X^*_+. \tag{2.2}$$

Let K be a compact space. The space $C(K)$ is a Banach lattice, it is a complete Banach lattice if K is a Stonian compact space. The L_p $(1 \leq p \leq \infty)$ are complete Banach lattices and any reflexive Banach lattice is a complete Banach lattice.

For the convenience of the reader, we give in this section some elementary definitions and fundamental properties relative to sublinear operators. For more details see [2], [13].

Definition 2.1. *A mapping T from a Banach space X into a Banach lattice Y is said to be sublinear if for all x, y in X and λ in $\mathbb{R}_+$, we have*

$$\begin{array}{rrcll} (i) & T(\lambda x) & = & \lambda T(x) & \text{(i.e., positively homogeneous)}, \\ (ii) & T(x+y) & \leq & T(x)+T(y) & \text{(i.e., subadditive)}. \end{array}$$

The operator T is said to be superlinear if T is positively homogeneous and superadditive (i.e., $T(x+y) \geq T(x) + T(y)$ for all x, y in X (i.e.,$-T$ is sublinear). Note that the sum of two sublinear operators is a sublinear operator and the multiplication by a positive number is also a sublinear operator. Let us denote by

$$\mathcal{SL}(X,Y) = \{\text{sublinear mappings } T : X \longrightarrow Y\ \}$$

and we equip it with the natural order induced by Y

$$T_1 \leq T_2 \Longleftrightarrow T_1(x) \leq T_2(x), \quad \forall x \in X. \tag{2.3}$$

We will note by ∇T the subdifferential of T, which is the set of all linear operators $u : X \longrightarrow Y$ such that $u(x) \leq T(x)$ for all x in X. We know (see [2]) that ∇T is not empty if Y is a complete Banach lattice and $T(x) = \sup\{u(x) : u \in \nabla T\}$, moreover, the supremum is attained (i.e., for all x in X there is $u_x \in \nabla T$ such that, $T(x) = u_x(x)$). If Y is simply a Banach lattice then ∇T is empty in general (see [13]). As a consequence

$$u \in \nabla T \Longleftrightarrow -T(-x) \leq u(x) \leq T(x), \quad \forall x \in X. \tag{2.4}$$

Proposition 2.1. *Let T be a sublinear from a Banach space X into a Banach lattice Y. Then, the following properties are equivalent.*

(1) *The operator T is continuous on X.*
(2) *The operator T is continuous in 0.*
(3) *There is a constant $C > 0$ such that for all $x \in X$, $\|T(x)\| \leq C\|x\|$.*
(4) *The operator T is Lipschitz and $\|T(x) - T(y)\| \leq K\|x - y\|$, for some positive constant K.*

Proof. The proof is in general like the linear case. For (3) implies (4), we have

$$|T(x) - T(y)| \leq |T(x-y)| + |T(y-x)|\,. \tag{2.5}$$

and this concludes the proof. □

In both cases, we say that T is bounded and we put

$$\|T\| = \sup\{\|T(x)\| : \|x\|_{\mathcal{B}_X} = 1\}.$$

Immediately, we have for all $x \in X$

$$\|T(x)\| \leq \sup_{u \in \nabla T} \|u(x)\| \leq \|T(x)\| + \|T(-x)\| \tag{2.6}$$

and consequently

$$\|T\| \leq \sup_{u \in \nabla T} \|u\| \leq 2\,\|T\|\,. \tag{2.7}$$

This gives the following result: let C be a positive constant, then the operator T is bounded and $\|T\| \leq C$ if, and only if, for all $u \in \nabla T$, $\|u\| \leq C$.

We denote also by $\mathrm{Lip}_0(X,Y)$ the Banach space of Lipschitz functions $T : X \to Y$ such that $T(0) = 0$ with pointwise addition and the Lipschitz norm $\mathrm{Lip}(.)$ is given by $\mathrm{Lip}(f) = \sup_{x\neq y} \frac{\|f(x) - f(y)\|}{\|x-y\|}$. We use the shorthand $X^{\#} := \mathrm{Lip}_0(X) := \mathrm{Lip}_0(X,\mathbb{R})$. The closed unit ball $\mathcal{B}_{X^{\#}}$ of $X^{\#}$ is a compact Hausdorff space for the topology of pointwise convergence on X. We will denote by $\mathcal{SB}(X,Y)$ the set of all bounded sublinear operators from X into Y. The set $\mathcal{SB}(X,Y)$ is a pointed positive convex cone of $\mathrm{Lip}_0(X,Y)$ but not salient and the great vector space in $\mathcal{SB}(X,Y)$ is

$$\mathcal{SB}(X,Y) \cap (-\mathcal{SB}(X,Y)) = \mathcal{B}(X,Y).$$

We can define a preorder on $\mathrm{Lip}_0(X,Y)$ by setting

$$T \leq S \text{ if } S - T \in \mathcal{SB}(X,Y).$$

Remark 2.1. *We have by Proposition 2.1, for $T \in \mathcal{SB}(X,Y)$*

$$\|T\| \leq \mathrm{Lip}(T) \leq 2\|T\|.$$

In addition if T is symmetric (i.e., $T(x) = T(-x)$ for all x in X), then

$$\|T\| = \mathrm{Lip}(T).$$

In the sequel, we can see $\mathcal{SB}(X,Y)$ as a cone in $(\mathrm{Lip}_0(X,Y), \mathrm{Lip}(.))$. We denote by

$$\triangle\mathcal{SB}(X,Y) = \mathcal{SB}(X,Y) - \mathcal{SB}(X,Y)$$

the subspace of $\mathrm{Lip}_0(X,Y)$ spanned by $\mathcal{SB}(X,Y)$, i.e.

$$\triangle\mathcal{SB}(X,Y) = \{T_1 - T_2 : T_1, T_2 \in \mathcal{SB}(X,Y)\}.$$

Remark 2.2. *For all T in $\triangle\mathcal{SB}(X,Y)$ there is $\varphi_T \in \mathcal{SB}(X,Y)$ and ψ_T super linear such that $\varphi_T \leq T \leq \psi_T$ and $\varphi_T(-x) = \varphi_{-T}(x)$ (resp. $\psi_T(-x) = \psi_{-T}(x)$) for all x in X. We define φ_T, ψ_T by*

$$\psi_T(x) = T_1(x) + T_2(-x) \quad \varphi_T(x) = -T_1(-x) - T_2(x).$$

where $T = T_1 - T_2$.

3 LIPSCHITZ p-SUMMING SUBLINEAR OPERATORS

We start by giving some standard notations. We denote by $\|.\|_p$ the norm on l_p of a sequence of real numbers. For a sequence of vectors $(x_i)_i$ in a Banach space X, its strong p-norm is the l_p-norm of the sequence $(\|x_i\|)_i$ and we denote its weak p-norm (cf. [9]) by

$$\omega_p((x_i)_i) = \sup_{x^* \in \mathcal{B}_{X^*}} \|(x^*(x_i))_i\|_p.$$

We denote respectively these spaces by $l_p(X)$ and $l_p^{\omega}(X)$ ($l_p^n(X)$ and $l_p^{n\,\omega}(X)$ if we take finite sequences $(x_i)_{1\leq i\leq n} \subset X$). We know (see [10]) that $l_p(X) \equiv l_p^{\omega}(X)$ (the symbol $\equiv$ indicates that two Banach spaces are isometrically isomorphic) for some $1 \leq p < \infty$ if, and only if, $\dim(X)$ is finite. If $p = \infty$, we have $l_\infty(X) \equiv l_\infty^{\omega}(X)$. We have also if $1 < p \leq \infty$, $l_p^{\omega}(X) \equiv \mathcal{B}(l_{p^*}, X)$ isometrically (where p^* is the conjugate of p *i.e.* $\frac{1}{p} + \frac{1}{p^*} = 1$). In other words, let $v : l_{p^*} \longrightarrow X$ be a linear operator such that $v(e_i) = x_i$ (namely $v = \sum_1^{\infty} e_j \otimes x_j$, e_j denotes the unit vector basis of l_p) then $\|v\| = \|(x_n)\|_{l_p^{\omega}(X)}$.

Analogously for a sequence $(\lambda_i)_i$ of real numbers and $(x_i)_i$, $(x_i')_i$ of points in X, we denote their weak Lipschitz p-norm (not really a norm because there is no linear structure) by

$$\omega_p^L\left((\lambda_i),(x_i)_i,(x_i')_i\right)=\sup_{f\in\mathcal{B}_{X^\#}}\left\|\left(|\lambda_i|^{\frac{1}{p}}\left(f(x_i)-f(x_i')\right)\right)_i\right\|_p. \tag{3.1}$$

Inspired by the useful concept of absolutely summing operators, J. Farmer and W. B. Johnson introduced in [11] the following definition. This is a good generalization of the concept of linear p-summing operators, since it is shown in [11] that the Lipschitz p-summing norm of a linear operator is the same as its p-summing norm.

Definition 3.1. *A Lipschitz operator T between X,Y is called Lipschitz p-summing $(1\leq p<\infty)$, if there is a positive constant C such that for all n in $\mathbb{N}$, $(x_i)_{1\leq i\leq n},(y_i)_{1\leq i\leq n}$ in X and $(a_i)_{1\leq i\leq n}$ in $\mathbb{R}^+$, we have*

$$\left\|\left(a_i^{\frac{1}{p}}\left(T(x_i)-T(y_i)\right)\right)\right\|_p\leq C\omega_p^L\left((a_i),(x_i)_i,(y_i)_i\right). \tag{3.2}$$

We denote by $\Pi_p^L(X,Y)$ the Banach space of all Lipschitz p-summing operators from X into Y $(1\leq p<\infty)$. The Lipschitz p-summing $(1\leq p<\infty)$ norm, $\pi_p^L(T)$ of T is the smallest constant C verifying (3.2).

Notice that for any embedding $j:Y\to Z$, we have $\pi_p^L(T)=\pi_p^L(jT)$ and

$$\pi_p^L(T)=\sup_{X_0\subset X}\left\{\pi_p^L(T/X_0):X_0\text{ finite dimensional subspace of }X\right\}.$$

Also, we can omit in the definition the sequence $(a_i)_{1\leq i\leq n}$ in $\mathbb{R}_+$ and we restrict to $a_i=1$ (see [11] for an implicit proof).

Now, we proceed to generalize (2.7) to the notion of Lipschitz p-summing operators, i.e., we study the following question: let C be a positive constant, then the operator $T\in\mathcal{SB}(X,Y)$ is Lipschitz p-summing and $\pi_p^L(T)\leq C$ if, and only if, for all $u\in\nabla T$, $\|\pi_p(u)\|\leq C$? This question is difficult for us, but we can resolve it partially by introducing a new intermediary notion of summability which we call "Lipschitz super p-summing operators". Also, we answer negatively a question posed in [2] in the linear case, i.e., if T is "p-summing" $1\leq p<\infty$ in the sense of definition below (Definition 3.2), then for all u in ∇T, u is p-summing and the reciprocal is false.

Definition 3.2. *A map T in $\Delta\mathcal{SB}(X,Y)$ is called Lipschitz super p-summing $(1\leq p<\infty)$, if there is a positive constant C such that for all n in $\mathbb{N}$, $(x_i)_{1\leq i\leq n},(y_i)_{1\leq i\leq n}$ in X and $(a_i)_{1\leq i\leq n}$ in $\mathbb{R}^+$, we have*

$$\left\|\left(a_i^{\frac{1}{p}}T(x_i-y_i)\right)\right\|_p\leq C\omega_p^L\left((a_i),(x_i)_i,(y_i)_i\right). \tag{3.3}$$

and for all x,y in X, $\|T(x-y)\|\leq Cd(x,y)$ if p is infinite.

We denote by $\Pi_p^{Ls}(X,Y)$ the space of the Lipschitz super p-summing $(1\leq p<\infty)$ in $\Delta\mathcal{SB}(X,Y)$ and by $\pi_p^{Ls}(T)$, the Lipschitz super p-summing norm of T; which is the smallest constant C verifying (3.3).

Notice that for any embedding $j:Y\to Z$, we have $\pi_p^{Ls}(T)=\pi_p^{Ls}(jT)$ and

$$\pi_p^{Ls}(T)=\sup_{X_0\subset X}\left\{\pi_p^{Ls}(T/X_0):X_0\text{ finite dimensional space of }X\right\}.$$

Also, we can omit in the definition the sequence $(a_i)_{1\leq i\leq n}$ in $\mathbb{R}_+$ and we restrict to $a_i=1$.

Remark 3.1. *By (2.5), if T is in $\Pi_p^{Ls}(X,Y)$ then T is in $\Pi_p^L(X,Y)$ and $\pi_p^L(T)\leq 2\pi_p^{Ls}(T)$. The converse is false see Example 1 below.*

Now we are ready to give the Pietsch domination theorem. The proof follows in an analogous way of [11, Theorem 1].

Theorem 3.1. *Let $1\leq p<\infty$. Let X be a Banach space and Y be a Banach lattice. The following properties are equivalent for T in $\Delta\mathcal{SB}(X,Y)$ and a positive constant C.*

(a) The mapping T is Lipschitz super p-summing and $\pi_p^{Ls}(T)\leq C$.

(b) There is a probability μ on $\mathcal{B}_{X^{\#}}$ such that

$$\|T(x-y)\| \leq C\left(\int_{\mathcal{B}_{X^{\#}}}|f(x)-f(y)|^{p}\, d\mu(f)\right)^{\frac{1}{p}} \tag{3.4}$$

for all x, y in X.

Proof. The proof is the same than that used in the Lipschitz p-summing case. □

As an immediate consequence, we have an inclusion result and some composition properties of our class.

Proposition 3.1. *Let $1 \leq p < q < \infty$. If $T : X \longrightarrow Y$ is Lipschitz super p-summing, then T is Lipschitz super q-summing and $\pi_{q}^{Ls}(T) \leq \pi_{p}^{Ls}(T)$.*

Proposition 3.2. *Let $S : E \longrightarrow X$ be a bounded linear function and T in $\Pi_{p}^{Ls}(X, Y)$. Then, $T \circ S$ is in $\Pi_{p}^{Ls}(E, Y)$ and $\pi_{p}^{Ls}(T \circ S) \leq \pi_{p}^{Ls}(T)\|S\|$.*

Proof. Let x, y be in X. Then

$$\begin{aligned}
& \|(T\circ S)(x-y)\| \\
= \;& \|T(S(x)-S(y))\| \\
\leq \;& \pi_{p}^{Ls}(T)\left(\int_{\mathcal{B}_{X^{\#}}}|f(S(x))-f(S(y))|^{p}\, d\mu(f)\right)^{\frac{1}{p}} \\
\leq \;& \pi_{p}^{Ls}(T)\|S\|\left(\int_{\mathcal{B}_{E^{\#}}}|g(x)-g(y)|^{p}\, d\mu(g)\right)^{\frac{1}{p}}.
\end{aligned}$$

Where $g(x) = \dfrac{f(S(x))}{\|S\|}$. Therefore, $T \circ S$ is Lipschitz super p-summing and

$$\pi_{p}^{Ls}(T\circ S) \leq \pi_{p}^{Ls}(T)\|S\|.$$

This ends the proof. □

Proposition 3.3. *Consider S in $\Pi_{p}^{Ls}(X, Y)$ and let u be in $\mathcal{B}^{+}(Y, Z)$ (Z is a Banach lattice). Then, $u \circ S$ is in $\Pi_{p}^{Ls}(X, Z)$ and $\pi_{p}^{Ls}(u \circ S) \leq \|u\|\, \pi_{p}^{Ls}(S)$.*

Proof. Let x, y be in X. Then

$$\begin{aligned}
\|(u\circ S)(x-y)\| &\leq \|u\|\,\|S(x-y)\| \\
&\leq \|u\|\,\pi_{p}^{Ls}(S)\left(\int_{B_{X^{\#}}}|f(x)-f(y)|^{p}\, d\mu(f)\right)^{\frac{1}{p}}.
\end{aligned}$$

This implies that, $u \circ S$ is Lipschitz super p-summing and $\pi_{p}^{Ls}(u \circ S) \leq \|T\|\, \pi_{p}^{Ls}(S)$. □

Proposition 3.4. *Let X be a Banach space and Y be a complete Banach lattice. Let T be a bounded sublinear operator from X into Y. Suppose that T is Lipschitz super p-summing ($1 \leq p < \infty$). Then for all $S \in \mathcal{SB}(X, Y)$ such that $S \leq T$, S is Lipschitz super p-summing.*

Proof. By (2.3) and (2.4), we have for all x, y in X, $S(x-y) \leq T(x-y)$ and $-S(x-y) \leq T(y-x)$. Thus $|S(x-y)| \leq |T(x-y)| + |T(x-y)|$. Using (3.4), we get $\pi_{p}^{Ls}(S) \leq 2\pi_{p}^{Ls}(T)$. □

Corollary 3.1. *If T is Lipschitz super p-summing ($1 \leq p < \infty$), then for all $u \in \nabla T$, u is p-summing.*

Let now $T : X \to Y$ be a sublinear operator between a Banach space X and a Banach lattice Y.

Definition 3.3. *[10, Theorem 2.18]We will say that T is "p-summing" ($1 \leq p < \infty$, we write $T \in \Pi_p(X, Y)$), if there exists a positive constant C such that for every n in $\mathbb{N}$ the mappings*

$$\begin{array}{lllll} T_n: & l_p^{n\,\omega}(X) & \longrightarrow & l_p^n(Y) & \\ & (x_i) & \longmapsto & \longmapsto & (T(x_i))_{1\leq i\leq n} \end{array}$$

are uniformly bounded by C. We put in this case

$$\pi_p(T) = \sup_n \|T_n\|.$$

If T is linear, then from the closed graph theorem, it is p-summing if, and only if, it satisfies that for every infinite sequence $(x_n)_{n\in\mathbb{N}}$ in X, we have

$$\left(\sum_{n\in\mathbb{N}} |\langle \xi, x_n\rangle|^p < \infty, \quad \forall \xi \in X^*\right) \Longrightarrow \sum_{n\in\mathbb{N}} \|T(x_n)\|^p < \infty. \tag{3.5}$$

Proposition 3.5. *Let u be a bounded linear operator from a Banach space X into a Banach lattice Y and $1 \leq p < \infty$. Then, the notions of Lipschitz super p-summing, Lipschitz p-summing and p-summing coincide and $\pi_p(u) = \pi_p^L(u) = \pi_p^{Ls}(u)$.*

Proof. We have $\pi_p(u) = \pi_p^L(u)$ by [11, Theorem 2] and $\pi_p^L(u) = \pi_p^{Ls}(u)$ because u is linear. □

By the weak Dvoretzky-Rogers Theorem which state that: every infinite dimensional Banach space X contains a weakly p-summable ($1 \leq p < \infty$) sequence which fails to be strongly p-summable (see [10, Theorem 2.18]). We then have the following corollary.

Corollary 3.2. *Let X be a Banach lattice. Then, the following properties are equivalent.*
(1) *The identity is in $\Pi_p^{Ls}(X, X)$*
(2) *The identity is in $\Pi_p^L(X, X)$.*
(3) *The identity is in $\Pi_p(X, X)$.*
(4) *The space X is of finite dimensional.*

Proof. The assertions (1) (2) and (3) are equivalent by Proposition 3.5. The assertion (2) is equivalent to (4) by the weak Dvoretzky-Rogers Theorem. □

We study in the Lipschitz case, the following question posed in [2]: Does u is Lipschitz p-summing (i.e., p-summing) for every u in ∇T if, and only if, T is p-summing sublinear operator (this question was inspired by 2.7)? (For the linear case, we have shown that, if T is p-summing then u is p-summing for every u in ∇T. The converse has remained open. In this paper, we ask this question and we prove that the reciprocal is false by the following example.

Example 1. *Let X be a Banach space of infinite dimensional and consider the map $T\colon X \longrightarrow \mathbb{R}$ defined by $T(x) = \|x\|$. The operator T is a bounded sublinear operator. The set*

$$\nabla T = \{u \in \mathcal{B}(X, \mathbb{R}) \subset \mathcal{B}_{X^*}, u \leq T\}$$

is in $\Pi_p(X, \mathbb{R})$ and $\pi_p(u) \leq \|T\|$. But T is not p-summing. Indeed, suppose the contrary, then there is an absolute positive constant C such that for all n in $\mathbb{N}$ and for all $(x_i)_{1\leq i\leq n}$ in X, we have

$$\sum_{i=1}^n \|x_i\|^p \leq C^p \sup_{\xi\in\mathcal{B}_{X^*}} \sum_{i=1}^n |\xi(x_i)|^p.$$

This implies that the Id_X is p-summing and by (3.5), we have for every infinite sequence $(x_i)_{1\leq i\leq n}$ in X

$$\left(\sum_{n\in\mathbb{N}} |\langle \xi, x_n\rangle|^p < \infty, \quad \forall \xi \in X^*\right) \Longrightarrow \sum_{n\in\mathbb{N}} \|T(x_n)\|^p < \infty.$$

This contradicts the weak Dvoretzky-Rogers Theorem.

Remark 3.2. *Let X be a Banach space and Y be a Banach lattice. The notions of p-summing and Lipschitz p-summing do not coincide on $\mathcal{SB}(X,Y)$. The opposite makes our problem trivial.*

Let X be a Banach space of infinite dimensional. The sublinear operator T defined on X by $T(x)=\|x\|$ is not p-summing by Example 1 but is Lipschitz p-summing. Indeed, for $(x_i)_{1\leq i\leq n}$, $(y_i)_{1\leq i\leq n}$in X, we have

$$\begin{aligned}
& \left(\sum_{i=1}^{n}|T(x_i)-T(y_i)|^p\right)^{\frac{1}{p}} \\
= & \left(\sum_{i=1}^{n}|\|x_i\|-\|y_i\||^p\right)^{\frac{1}{p}} \\
= & \left(\sum_{i=1}^{n}|f_0(x_i)-f_0(y_i)|^p\right)^{\frac{1}{p}} \\
& \text{(where } f_0(.)=\|.\|\text{; which is a Lipschitz function and Lip}(f_0)\leq 1) \\
\leq & \sup_{f\in\mathcal{B}_{X^\#}}\left(\sum_{i=1}^{n}|f(x_i)-f(y_i)|^p\right)^{\frac{1}{p}}.
\end{aligned}$$

We can add that the sublinear operator T is not super Lipschitz p-summing. In fact, suppose that there exists $C>0$ such that for every $(x_i)_{1\leq i\leq n}$, $(y_i)_{1\leq i\leq n}$in X, we have

$$\begin{aligned}
& \left(\sum_{i=1}^{n}|T(x_i-y_i)|^p\right)^{\frac{1}{p}} \\
= & \left(\sum_{i=1}^{n}\|x_i-y_i\|^p\right)^{\frac{1}{p}} \\
\leq & C\sup_{\mathcal{B}_{X^\#}}\left(\sum_{i=1}^{n}|f(x_i)-f(y_i)|^p\right)^{\frac{1}{p}}.
\end{aligned}$$

This implies that Id_X is Lipschitz p-summing and consequently by Corollary 3.2 is p-summing. This contradicts the weak Dvoretzky-Rogers theorem.

The following proposition studies the link between $u\in\nabla T$ and T, for T Lipschitz super p-summing.

Proposition 3.6. *Let X be an arbitrary Banach space and Y be a complete Banach lattice. Consider T in $\mathcal{SB}(X,Y)$. For $1\leq p<\infty$, if T is in $\Pi_p^{Ls}(X,Y)$, then $\nabla T\subset\Pi_p^{Ls}(X,Y)$. The converse is false in general.*

Proof. Let T be in $\Pi_p^{Ls}(X,Y)$. We have

$$\begin{aligned}
\|u(x-y)\| &\leq \|T(x-y)\|+\|T(y-x)\| \\
&\leq 2\pi_p^{Ls}(T)\left(\int_{\mathcal{B}_{X^\#}}|f(x)-f(y)|^p\,d\mu(f)\right)^{\frac{1}{p}}.
\end{aligned}$$

Hence u is $\Pi_p^{Ls}(X,Y)=\Pi_p(X,Y)$ and $\pi_p(u)\leq 2\pi_p^{Ls}(T)$. According to the previous result the opposite is false.
Consider $T\colon X\longrightarrow\mathbb{R}$ defined $T(x)=\|x\|$. Then T is a bounded sublinear operators. We have

$$\nabla T=\left\{\begin{array}{c} u\in\mathcal{B}(X,\mathbb{R})\subset \\ X^*,\quad u\leq T\end{array}\right\}\subset\Pi_p(X,\mathbb{R})\subset\Pi_p^{Ls}(X,\mathbb{R}).$$

Thus $\nabla T\subset\Pi_p^{Ls}(X,\mathbb{R})$ but T is not Lipschitz super p-summing. Suppose that T is Lipschitz super p-summing, then there is a positive constant C such that for every n in $\mathbb{N}$, $(x_i)_{1\leq i\leq n},(y_i)_{1\leq i\leq n}$ in X; we have

$$\begin{aligned}\left(\sum_{i=1}^{n}|T(x_i-y_i)|^p\right)^{\frac{1}{p}} &= \left(\sum_{i=1}^{n}\|x_i-y_i\|^p\right)^{\frac{1}{p}}\\ \text{(by Corollary 3.2)} &\leq \sup_{f\in\mathcal{B}_{X^\#}}\|(f(x_i)-f(y_i))_i\|_p .\end{aligned}$$

This implies that $\dim(X)$ is finite and hence contradiction. □

Problem 1. *For which spaces X, Y we have $\Pi_p^{Ls}(X,Y)=\Pi_p^L(X,Y)$?*

Problem 2. *Let X be a Banach lattice (for example $X=L_q$). Is the Lipschitz operator*

$$\begin{array}{cccc} T: & X & \longrightarrow & X\\ & x & \longmapsto & T(x)=|x| \end{array}$$

Lipschitz p-summing for some p?

4 Relationship between T and $u\in\nabla T$ for other types of summability

We give now the notion of Lipschitz p-dominated operators introduced by D. Chen and B. Zheng in [5]. A Lipschitz mapping $T:X\to Y$ between Banach spaces X and Y is Lipschitz p-dominated ($1\leq p<\infty$) if there exist a Banach space Z and a linear operator L in $\Pi_p(X,Z)$ such that

$$\| T(x)-T(y)\|\leq\| L(x-y)\|,\quad \forall x,y\in X. \tag{4.1}$$

The class of all Lipschitz p-dominated operators between X and Y is denoted by $\mathcal{D}_p^L(X,Y)$. For T in $\mathcal{D}_p^L(X,Y)$, we set $\mathrm{d}_p^L(T)$ to be the infimum of $\pi_p(L)$, the infimum being taken over all the above Z and L. If $T(0)=0$, we have the following reformulation

$$\| T(x-y)\|\leq\| L(x-y)\|,\quad \forall x,y\in X.$$

Indeed, if we replace in (4.1) x by $x-y$ and y by 0, we will have $\| T(x-y)\|\leq\| L(x-y)\|,\quad \forall x,y\in X$. The converse is true for sublinear operators, by (2.5) we have $\| T(x)-T(y)\|\leq\| 2L(x-y)\|,\quad \forall x,y\in X$.

Proposition 4.1. *Let X be an arbitrary Banach space and Y be a complete Banach lattice. Consider T in $\triangle\mathcal{SB}(X,Y)$. If T is in $\mathcal{D}_p^L(X,Y)$ for $1\leq p<\infty$ then, T is p-summing in the sense of Definition 3.3 and hence T is Lipschitz super p-summing, which implies that T is Lipschitz p-summing.*

Proof. From [5, Theorem 3.2], there is a regular Borel probability measure μ on $\mathcal{B}_{X^*}$ such that for all $x,y\in X$, we have

$$\|T(x)-T(y)\|\leq \mathrm{d}_p^L(T)\left(\int_{\mathcal{B}_{X^*}}|\xi(x)-\xi(y)|^p\,d\mu(\xi)\right)^{\frac{1}{p}}. \tag{4.2}$$

If we take $y=0$, we obtain $\|T(x)\|\leq \mathrm{d}_p^L(T)\left(\int_{\mathcal{B}_{X^*}}|\xi(x)|^p\,d\mu(\xi)\right)^{\frac{1}{p}}$. Replace in the precedent inequality x by $x-y$, we get

$$\begin{aligned}\|T(x-y)\| &\leq \mathrm{d}_p^L(T)\left(\int_{\mathcal{B}_{X^*}}|\xi(x-y))|^p\,d\mu(\xi)\right)^{\frac{1}{p}},\\ &\leq \mathrm{d}_p^L(T)\left(\int_{\mathcal{B}_{X^*}}|\xi(x)-\xi(y)|^p\,d\mu(\xi)\right)^{\frac{1}{p}}.\end{aligned}$$

By using Remark 3.1, we finish the last implication. □

Proposition 4.2. *Let X be an arbitrary Banach space and Y be a complete Banach lattice. Consider T in $\mathcal{SB}(X,Y)$. Then, if T is in $\mathcal{D}_p^L(X,Y)$ for $1\leq p<\infty$, we have $\nabla T\subset\mathcal{D}_p^L(X,Y)$ and consequently $\nabla T\subset\Pi_p^L(X,Y)$.*

Proof. Using (2.5) and (2.6), we have for all u in ∇T

$$\|u(x-y)\| \leq \|T(x-y)\| + \|T(y-x)\|$$

and hence by (4.2)

$$\begin{aligned} \|u(x-y)\| &\leq 2\mathrm{d}_p^L(T)\left(\int_{\mathcal{B}_{X^*}} |\xi(x-y))|^p \, d\mu\left(\xi\right)\right)^{\frac{1}{p}}, \\ \text{(because } T(0)=0) &\leq 2\mathrm{d}_p^L(T)\left(\int_{\mathcal{B}_{X^*}} |\xi(x)-\xi(y))|^p \, d\mu\left(\xi\right)\right)^{\frac{1}{p}}. \end{aligned}$$

This implies that u is p-summing and consequently is Lipschitz p-summing. □

The definition of "*Lipschitz strongly p-summing operators*" was introduced independently by [19] and [20]. We deduce in the same spirit from that used in, [12] the following definition.

Definition 4.1. *A Lipschitz map $T : X \to E$ between a Banach space X and a Banach lattice E is latticially Lipschitz strongly p-summing $(1 < p \leq \infty)$ if there is a constant $C > 0$, such that for all $n \in \mathbb{N}$, $(x_i)_{1\leq i\leq n}$, $(x_i')_{1\leq i\leq n}$ in X, $(y_i^*)_{1\leq i\leq n}$ in E_+^* and $(\lambda_i)_{1\leq i\leq n}$ in $\mathbb{R}_+$, we have*

$$\sum_{i=1}^{n}\lambda_i\left|\left\langle T\left(x_i\right)-T\left(x_i'\right),y_i^*\right\rangle\right| \leq C\left(\sum_{i=1}^{n}\lambda_i\left\|x_i-x_i'\right\|^p\right)^{\frac{1}{p}}\sup_{y^{**}\in\mathcal{B}_{E^{**}}^+}\|(\langle y_i^*,y^{**}\rangle)\|_{l_{p^*}^n} \tag{4.3}$$

If T is sublinear, then this definition is equivalent by (2.5) to

$$\sum_{i=1}^{n}\lambda_i\left|\left\langle T\left(x_i-x_i'\right),y_i^*\right\rangle\right| \leq C\left(\sum_{i=1}^{n}\lambda_i\left\|x_i-x_i'\right\|^p\right)^{\frac{1}{p}}\sup_{y^{**}\in\mathcal{B}_{E^{**}}^+}\|(\langle y_i^*,y^{**}\rangle)\|_{l_{p^*}^n}$$

We denote by $\mathcal{D}_{st,p}^{+L}(X,E)$ the class of all latticially Lipschitz strongly p-summing operators from X into E and $d_{st,p}^{+L}(T)$ the smallest C such that (4.3) holds. This generalizes the definition introduced by [8] in the linear case. If T is linear, then we have $\mathcal{D}_{st,p}^{L}(X,E) = \mathcal{D}_p(X,E)$ because $\mathcal{B}_{X^\#}$ is not involved in the definition.

Remark 4.1. *Blasco in [3] introduces this definition under the name of positive p-summing but the sup of second member is taken on all the ball $\mathcal{B}_{E^{**}}$. This two definitions are the same.*

Indeed, we have for a Banach lattice X and $(x_i)_{1\leq i\leq n} \subset X^+$

$$\begin{aligned} \sup_{x^*\in\mathcal{B}_{X^{\cdot*}}}\left(\sum_{i=1}^{n}|\langle x_i,x^*\rangle|^p\right)^{\frac{1}{p}} &= \sup_{x^*\in\mathcal{B}_{X^*}}\sup_{(\alpha_i)\in\mathcal{B}_{l_{p^*}}}\left(\left|\sum_{i=1}^{n}\alpha_i\langle x_i,x^*\rangle\right|\right) \\ &= \sup_{x^*\in\mathcal{B}_{X^*}}\sup_{(\alpha_i)\in\mathcal{B}_{l_{p^*}}}\left(\left|\left\langle\sum_{i=1}^{n}\alpha_i x_i,x^*\right\rangle\right|\right) \\ &= \sup_{(\alpha_i)\in\mathcal{B}_{l_{p^*}}^+}\sup_{x^*\in\mathcal{B}_{X^*}}\left(\left|\left\langle\sum_{i=1}^{n}\alpha_i x_i,x^*\right\rangle\right|\right) \\ &= \sup_{(\alpha_i)\in\mathcal{B}_{l_{p^*}}}\sup_{x^*\in\mathcal{B}_{X_+^*}}\left(\left|\left\langle\sum_{i=1}^{n}\alpha_i x_i,x^*\right\rangle\right|\right) \\ &= \sup_{x^*\in\mathcal{B}_{X_+^*}}\left(\sum_{i=1}^{n}|\langle x_i,x^*\rangle|^p\right)^{\frac{1}{p}}. \end{aligned}$$

Let $T \in \mathrm{Lip}_0(X;E)$ and $v : l_p^n \to E^*$ be a linear operator ($\Longrightarrow$bounded). The Lipschitz operator T is strongly Lipschitz p-summing if, and only if,

$$\sum_{i=1}^{n}\lambda_i \left|\left\langle T\left(x_i\right)-T\left(x_i'\right), v(e_i)\right\rangle\right| \leq C\left(\sum_{i=1}^{n}\lambda_i \left\|x_i-x_i'\right\|^p\right)^{\frac{1}{p}}\|v\|.$$

This is equivalent (see [19] and [20]) to Pietsch's domination theorem; which is: there exist a constant $C>0$ and a Radon probability μ on $\mathcal{B}_{E^{**}}$ such that for all $x, x' \in X$ and $y^* \in E^*$, we have

$$\left|\left\langle T(x)-T\left(x'\right), y^*\right\rangle\right| \leq C\left\|x-x'\right\|\left\|y^*\right\|_{L_p(\mathcal{B}_{E^{**}},\mu)} \tag{4.4}$$

Moreover, in this case

$$d_{st,p}^{L}(T)=\inf\{C>0: \text{ for all } C \text{ verifying (4.4)}\}.$$

We give the following definition introduced in [1] and its domination theorem.

Definition 4.2. *Let $1 \leq p \leq +\infty$. A linear operator $u: X \longrightarrow Y$ between a Banach space X and a Banach lattice Y is positive strongly p-summing if there exists a constant $C>0$, such that for all finite sets $(x_i)_{i=1}^n \subset X$ and $(y_i^*)_{i=1}^n \subset Y_+^*$, we have*

$$\sum_{i=1}^{n}\left|\left\langle u\left(x_i\right), y_i^*\right\rangle\right| \leq C\left(\sum_{i=1}^{n}\|x_i\|^p\right)^{\frac{1}{p}} \sup_{y^{**}\in\mathcal{B}_{Y^{**}}^{+}}\left(\sum_{i=1}^{n}\left\langle y_i^*, y^{**}\right\rangle^{p^*}\right)^{\frac{1}{p^*}}. \tag{4.5}$$

Where $\mathcal{B}_{Y^{**}}^{+}=\{y^{**}\in\mathcal{B}_{Y^{**}}: y^{**}\geq 0\}=\mathcal{B}_{Y^{**}}\cap Y_+^{**}$.

The class of all positive strongly p-summing operators between X and Y is denoted by $\mathcal{D}_p^+(X,Y)$. The infimum of all the constant C in (4.5) defines the norm d_p^+ on $\mathcal{D}_p^+(X,Y)$. We have $\mathcal{D}_1^+(X,Y)=\mathcal{B}(X,Y)$.

The reformulation in continuous term by [1, Theorem 4.13] is : there exists a positive constant $C>0$ and Radon probability measure μ on $\mathcal{B}_{Y_+^{**}}$ such that for all $x \in X$ and $y^* \in Y^*$, we have

$$|\langle u(x), y^*\rangle| \leq C\|x\|\left(\int_{\mathcal{B}_{Y^{**}}^{+}}\left||y^*|\left(y^{**}\right)\right|^{p^*}d\mu\left(y^{**}\right)\right)^{\frac{1}{p^*}}. \tag{4.6}$$

Moreover, in this case

$$\mathrm{d}_p^+(u)=\inf\{C, \text{satisfying 4.6)}\}.$$

Proposition 4.3. *Let X be an arbitrary Banach space and Y be a complete Banach lattice. Consider T in $\mathcal{SB}(X,Y)$. Then, if T is in $\mathcal{D}_{st,p}^{L}(X,Y)$ for $1<p\leq\infty$, we have u positive strongly p-summing for all u in ∇T and hence u^* is positive p^*-summing and $\pi_{p^*}^{+}(u^*)\leq 2d_{st,p}^{L}(T)$.*

Proof. We have by (2.2) $\langle u(x), y^*\rangle \leq \langle T(x), y^*\rangle$ and $\langle u(-x), y^*\rangle \leq \langle T(-x), y^*\rangle$ for all y^* in Y_+^*. This implies that $|\langle u(x), y^*\rangle| \leq |\langle T(x), y^*\rangle| + \langle T(-x), y^*\rangle$. And hence by using (4.6)

$$|\langle u(x), y^*\rangle| \leq 2d_{st,p}^{L}(T)\|x\|\left(\int_{\mathcal{B}_{Y^{**}}}\left|y^*\left(y^{**}\right)\right|^{p^*}d\mu\left(y^{**}\right)\right)^{\frac{1}{p^*}}$$

for all y^* in Y_+^*. Consider $(x_i)_{i=1}^n \subset X$ and $(y_i^*)_{i=1}^n \subset Y^*$. By using Hölder inequality, we have

$$
\begin{aligned}
& \sum_{i=1}^{n} \left|\left\langle u\left(x_i\right), y_i^{*}\right\rangle\right| \\
\leq \quad & 2 d_{st,p}^{L}(T) \sum_{i=1}^{n} \|x_i\| \left(\int_{\mathcal{B}_{Y^{**}}} \left|y_i^{*}\left(y^{**}\right)\right|^{p^{*}} d\mu\left(y^{**}\right) \right)^{\frac{1}{p^{*}}} \\
\leq \quad & 2 d_{st,p}^{L}(T) \left(\sum_{i=1}^{n} \|x_i\|^{p} \right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} \int_{\mathcal{B}_{Y^{**}}} \left|y_i^{*}\left(y^{**}\right)\right|^{p^{*}} d\mu\left(y^{**}\right) \right)^{\frac{1}{p^{*}}} \\
\leq \quad & 2 d_{st,p}^{L}(T) \left(\sum_{i=1}^{n} \|x_i\|^{p} \right)^{\frac{1}{p}} \left(\sup_{y^{**} \in \mathcal{B}_{Y^{**}}} \sum_{i=1}^{n} \int_{\mathcal{B}_{Y^{**}}} \left|y_i^{*}\left(y^{**}\right)\right|^{p^{*}} d\mu\left(y^{**}\right) \right)^{\frac{1}{p^{*}}} \\
\leq \quad & 2 d_{st,p}^{L}(T) \left(\sum_{i=1}^{n} \|x_i\|^{p} \right)^{\frac{1}{p}} \left(\sup_{y^{**} \in \mathcal{B}_{Y^{**}}} \sum_{i=1}^{n} \left|y_i^{*}\left(y^{**}\right)\right|^{p^{*}} \right)^{\frac{1}{p^{*}}} \\
\leq \quad & 2 d_{st,p}^{L}(T) \left(\sum_{i=1}^{n} \|x_i\|^{p} \right)^{\frac{1}{p}} \left(\sup_{y^{**} \in \mathcal{B}_{Y^{**}}^{+}} \sum_{i=1}^{n} \left(y_i^{*}\left(y^{**}\right)\right)^{p^{*}} \right)^{\frac{1}{p^{*}}}.
\end{aligned}
$$

We immediately have u positive strongly p-summing from (4.5) and therefore $\mathrm{d}_p^{+}(u) \leq 2 d_{st,p}^{L}(T)$. The characterization (4.6) of positive strongly p-summing linear operators yields for all $x \in X$ and $y^{*} \in Y^{*}$ that

$$
\left|\left\langle u(x), y^{*}\right\rangle\right| \leq 2 d_{st,p}^{L}(T) \|x\| \left(\int_{\mathcal{B}_{Y^{**}}^{+}} \left(\left|y^{*}\right|\left(y^{**}\right)\right)^{p^{*}} d\mu\left(y^{**}\right) \right)^{\frac{1}{p^{*}}}
$$

and thus

$$
\left|\left\langle x, u^{*}\left(y^{*}\right)\right\rangle\right| \leq 2 d_{st,p}^{L}(T) \|x\| \left(\int_{\mathcal{B}_{Y^{**}}^{+}} \left(\left|y^{*}\right|\left(y^{**}\right)\right)^{p^{*}} d\mu\left(y^{**}\right) \right)^{\frac{1}{p^{*}}}.
$$

Hence

$$
\left\|u^{*}\left(y^{*}\right)\right\| \leq 2 d_{st,p}^{L}(T) \left(\int_{\mathcal{B}_{Y^{**}}^{+}} \left(\left|y^{*}\right|\left(y^{**}\right)\right)^{p^{*}} d\mu\left(y^{**}\right) \right)^{\frac{1}{p^{*}}}.
$$

This implies by [1, Proposition 3.4] that u^{*} is positive p^{*}-summing and $\pi_{p^{*}}^{+}\left(u^{*}\right) \leq 2 d_{st,p}^{L}(T)$. $\square$

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The authors declare no conflict of interest.

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